10/26 Isabelle Gallagher

Hierarchies:

$f_N(t, Z_N)$; $Z_N (z_1, \cdots, z_N)$; $z_j = (x_j, v_j) \in \mathbb{T}^2 \times \mathbb{R}^2$

$\partial_t f_N + V_N \cdot \nabla_{X_N} f_N = 0$ in

$$\mathcal{D}_\varepsilon^N = \{ \underset{\in \mathbb{T}^2 \times \mathbb{R}^2}{Z_N} \mid \underset{i \neq j}{|x_j - x_i| > \varepsilon} \}$$

$$f_N^{(1)}(t, z_1) = \int f_N(t, Z_N)\, dz_2 \cdots dz_N$$

BBGKY:

$$\partial_t f_N^{(s)} + V_s \nabla_{X_s} f_N^{(s)} = \alpha \underset{\alpha'}{C_{s,s+1}} f_N^{(s+1)}$$

$$\sum_{i=1}^{s} \int_{x_{s+1} = x_i + \varepsilon \nu} (N-s)\varepsilon^{d-1} f_N^{(s+1)} ((v_{s+1} - v_i) \cdot \nu)\, d\nu\, dv_{s+1}$$

$$f_N^{(1)}(t) = \sum_{n=0}^{N-1} \alpha^n \int\int dt_2 \cdots dt_{n+1} (S_1(t-t_2) C_{1,2} S_2(t_2 - t_3) \cdots f_N^{(n+1)}(0).$$

Boltzmann hierachy $N\varepsilon = \alpha$

$$f_\alpha^{(1)}(t) = \sum_{n \geq 0} \alpha^n \int \cdots \int S_1^0(t - t_2) C_{1,2}^0 \cdots f_{\alpha,0}^{(n+1)}(0).$$

Lanford:

$$f_N^{(s)}(t) \longrightarrow f_\alpha^{(s)}(t) \qquad \forall t \leq \frac{c}{\alpha}$$

outside diagonals.

$$\| f_N^{(s)}(t) \| := \sup_{Z_s} e^{\mu(t) s} e^{\frac{\beta(t)}{2} |V_s|^2} | f_N^{(s)}(t, Z_s) |$$

$$\mu(t) = \mu_0 - c\alpha t; \quad \beta(t) = \beta_0 - c\alpha t$$

<u>Choise of the initial data.</u>

<u>Boltzmann</u>: $f_N^0(Z_N) = \prod_{i=1}^{N} f_0(z_i) (M^{\otimes N}(V_N)) \mathbb{1}_{\mathcal{D}_\varepsilon^N} \mathcal{Z}_N^{-1}$

$$f_{\alpha,0}^N(Z_N) = \prod_{i=1}^{N} f_0(z_i) (M^{\otimes N}(V_N))$$

Linear Boltzmann: $f_N^0(Z_N) = M^{\otimes N}(V_N)\, g_0(x_1)\, \mathbb{1}_{D_\varepsilon^N}\, \mathcal{Z}_N^{-1}$

well-posed in L^∞_β $\qquad f_{\alpha,0}(Z_N) = \cdots\cdots$

Linearized Boltzmann. well-posed in L^2_β $\quad$ β means Gaussian decay

$$\partial_t g_\alpha + v\cdot \nabla_x g_\alpha = \alpha \int M(v_1')g(v') + M(v')g(v_1') - M(v_1)g(v_1) - M(v)g(v_1)$$

$$f_N^0(Z_N) = M^{\otimes N}(V_N) \sum_{i=1}^N g_{\alpha,0}(z_i), \quad \int M(v)\, g_{\alpha,0}(z)\, dz = 0$$

Thm $\quad d=2 \quad \|g_{\alpha,0}\|_{W^{1,\infty}} \le \exp(C\alpha^2) \qquad \beta = 1$

Assume $g_{\alpha,0}(z)$ converges to $\rho_0(x) + u_0(x)\cdot v + (|v|^2-2)\theta_0(x)$

Then $N\to\infty \quad N\varepsilon = \alpha \quad \| f_N^{(1)}(t) - M g_\alpha(t)\|_{L^2} \le C \dfrac{T^2 \exp C\alpha^2}{\log\log N}$

and $g(t) = \rho(t,x) + u(t,x)\, v + (|v|^2-2)\,\theta(t,x)$

$g_\alpha(t) \xrightarrow[\alpha\to\infty]{} g(t) \quad \alpha \sim (\log^{\otimes 3} N)^{1/2}$ $\Big|$ $\begin{cases} \partial_t \rho + \operatorname{div} u = 0, & \partial_t u + \nabla_x(\rho+\theta) = 0 \\ \partial_t \theta + \operatorname{div} u = 0, & \text{Acoustics} \end{cases}$

$g_\alpha(\alpha\tau) \xrightarrow[\alpha\to\infty]{} g(\tau) \qquad g_0(x) = u_0 v + (|v|^2-2)\theta_0(x), \quad \operatorname{div} u_0 = 0$

$$\partial_\tau u - C_0 \Delta u = 0, \quad \operatorname{div} u = 0, \quad \partial_\tau \theta - \mu_\beta \Delta\theta = 0$$

Differences with Lanford and linear settings.

1. Functional setting L^2 instead of L^∞
2. Size of the initial data in L^∞ $O(N)$ x worse.

Initial data L^2 norm?

$$\int \frac{(f^{\otimes N}(Z_N))^2}{M^{\otimes N}(V_N)}\, dZ_N = \int M^{\otimes N} \left(\Sigma g_{\alpha,0}(z_i)\right)^2 dZ_N \,(\mathcal{Z}_N^{-1})^2$$

$$= \sum_{i=1}^N \int M^{\otimes N}(V_N)\, g_{\alpha,0}^2(z_i)\, dZ_N\, (\mathcal{Z}_N^{-1})^2$$

$$\le N \|g_{\alpha,0}\|^2_{L^2_\beta} (\mathcal{Z}_N^{-1})^2$$

$\mathcal{Z}_N = \int \mathbb{1}_{D_\varepsilon^N}(X_N)\, dX_N$ $\qquad d=2$

$\mathcal{Z}_N^{-1} \le \exp(\underbrace{C\varepsilon\alpha N}_{C\alpha^2}) \qquad \mathcal{Z}_N^{-1} \mathcal{Z}_{N-s} \le (1 - C\varepsilon\alpha)^{-s}$

$f_{N,0}$ in $L^2_\beta \sim O(\sqrt{N})$, $f_{N,0}^{(s)} \sim O(\sqrt{s})$.

solution ? same size $f_N^{(s)}(t) \sim O(\sqrt{N})$!

Functional setting

$$\int \| \underbrace{C_{s,s+1} f_N^{(s+1)}}_{S_{s+1}(\tau)} \|_{L^2_\beta} d\tau \leq \| f_N^{(s+1)} \|_{L^2_\beta}$$

trace on $|x_i - x_{i+1}| = \varepsilon$

$$(z_s, \nu, \tau, v_{s+1}) \xrightarrow{S^0} Z_{s+1}(\tau) = (x_s - \tau V_s, V_s, x_i + \varepsilon\nu - \tau v_{s+1}, v_{s+1})$$

$$dz_{s+1} \leftrightarrow \varepsilon\, d\nu\, dv_{s+1}\, dz_s\, d\tau\, (v_{s+1} - v_i)\cdot\nu$$

L^2 $(N-\varepsilon)\varepsilon^{1/2}$ $\| C_{s,s+1} S^0_{s+1}(\tau) f_N^{(s+1)} \|_{L^2_\beta} \gtrsim \frac{1}{\sqrt{\varepsilon}}$

initial data, bounds; free flow !

Boltzmann hierarchy.

Free flow: $f_\alpha^{(s)}(t) = M^{\otimes s}(V_s) \sum_{i=1}^{s} g_\alpha(t, z_i)$

$$C_{s,s+1} S^0_{s+1}(\tau) f_\alpha^{(s+1)} = \int f_\alpha^{(s+1)}(t, z_s, x_i, v_{s+1})$$

$$= M^{\otimes s} \int g_\alpha(t, x_i, v_{s+1}). \rightarrow \text{No trace}$$

L^2 OK.

Proof of the theorem

$$f_N^{(1)}(t) = f_N^{(1,k)}(t) + R_N^{(k)}(t)$$

$\sim$ Lanford $< 2^k$ collisions on each time step.

"stop in L^2"

- no recollisions OK in L^2 (if $\varepsilon^{-1/2}$ dealt with)
- recollisions N

1 recollisions $\varepsilon |\log \varepsilon|^3$

2 recollisions ε $d=2$

Geometry: avoid recollisions

$f_{N,0}^{(J_k)}(z_{J_k})$

f_N symmetric function mean free

$$f_N(Z_N) = M^{\otimes N}(V_N) \sum_{i=1}^{N} \sum_{m \in \sigma_N^m} g_N^m(Z_\sigma)$$

with

$$\|g_N^m\|_{L^2_\beta}^2 \leq \frac{\exp(C\alpha^2)}{C_N^m} \|f_N\|_{L^2_\beta}^2$$

$m=1$: $\frac{1}{\sqrt{N}} \sim \sqrt{N}$