Dispersive analysis for Capillary-gravity Water waves in $3D$

Benoit Pausader
(with Y. Deng, A. Ionescu and F. Pusateri)

November 2, 2015

BROWN

Figure: "Diving grebe" by Brocken Inaglory. Licensed under CC BY-SA 3.0 via Commons - https://commons.wikimedia.org/wiki/File:Diving_grebe.jpg#/media/File :

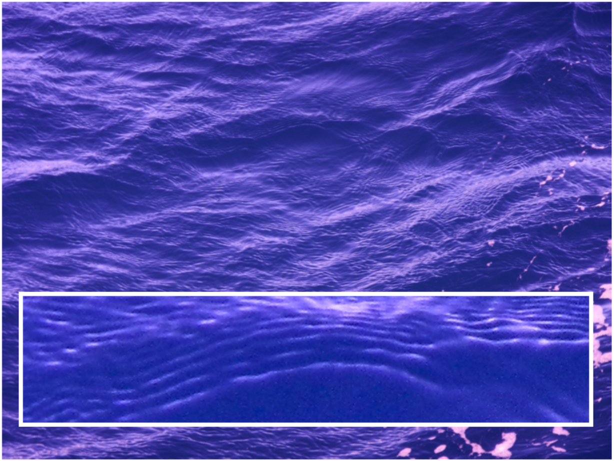


Figure: "Capillary 1" downloaded from http://epod.usra.edu/blog/2014/08/capillary-waves.html

Outline

Outline

1 Introduction

2 Dispersive analysis
- Quadratic space-time resonances
- Nonlinear analysis
  - properties of the phase

Setup

We consider the dynamics of an interface between an inert atmosphere (without dynamics) and a large body of incompressible, inviscid, irrotational water, subjected to both gravity and surface tension.

Setup

We consider the dynamics of an interface between an inert atmosphere (without dynamics) and a large body of incompressible, inviscid, irrotational water, subjected to both gravity and surface tension.

We consider a small perturbation of a fluid at rest (basic equilibrium) and we study its asymptotic behavior:

- does it converge back to equilibrium?

Setup

We consider the dynamics of an interface between an inert atmosphere (without dynamics) and a large body of incompressible, inviscid, irrotational water, subjected to both gravity and surface tension.

We consider a small perturbation of a fluid at rest (basic equilibrium) and we study its asymptotic behavior:

- does it converge back to equilibrium?
- does it lead to concentration of energy (and eventual blow-up?)

Setup

We consider the dynamics of an interface between an inert atmosphere (without dynamics) and a large body of incompressible, inviscid, irrotational water, subjected to both gravity and surface tension.

We consider a small perturbation of a fluid at rest (basic equilibrium) and we study its asymptotic behavior:

- does it converge back to equilibrium?
- does it lead to concentration of energy (and eventual blow-up?)

Setup

We consider the dynamics of an interface between an inert atmosphere (without dynamics) and a large body of incompressible, inviscid, irrotational water, subjected to both gravity and surface tension.

We consider a small perturbation of a fluid at rest (basic equilibrium) and we study its asymptotic behavior:

- does it converge back to equilibrium?
- does it lead to concentration of energy (and eventual blow-up?)

We show that the former is true (work with Y. **Deng**, A. **Ionescu** and F. **Pusateri**).

Equations

Hypothesis:

- Simple dynamics in the bulk: No dynamics in the air: $p \equiv Cte \equiv 0$, inviscid, incompressible, irrotational fluid in the water:

$$(\partial_t + v \cdot \nabla)\, v + \nabla p = -g e_y, \qquad \mathrm{div}(v) = 0 = \mathrm{curl}(v)$$

Equations

Hypothesis:

- Simple dynamics in the bulk: No dynamics in the air: $p \equiv Cte \equiv 0$, inviscid, incompressible, irrotational fluid in the water:

$$(\partial_t + v \cdot \nabla) v + \nabla p = -g e_y, \qquad \mathrm{div}(v) = 0 = \mathrm{curl}(v)$$

- Coupling by boundary conditions:

$$\begin{aligned} \text{(rest at } \infty) &: |u| \to 0, \qquad |x| \to \infty, \\ \text{cont. of stress tensor} &: [\![p]\!]\mathbf{n} = \sigma H \mathbf{n}, \\ \text{interface advected} &: (\partial_t + v_x \cdot \nabla_x) h = v_y. \end{aligned}$$

Model limitations

Most dynamical aspects of small waves remain poorly understood. Among them:

- Influence of the vorticity?

Model limitations

Most dynamical aspects of small waves remain poorly understood. Among them:

- Influence of the vorticity?
- Influence of the bottom (flat/periodic?)

Model limitations

Most dynamical aspects of small waves remain poorly understood. Among them:

- Influence of the vorticity?
- Influence of the bottom (flat/periodic?)
- Influence of the air? (e.g. formation of waves, see **Bühler-Shatah-Walsh-Zeng**)

Model limitations

Most dynamical aspects of small waves remain poorly understood. Among them:

- Influence of the vorticity?
- Influence of the bottom (flat/periodic?)
- Influence of the air? (e.g. formation of waves, see **Bühler-Shatah-Walsh-Zeng**)

Model limitations

Most dynamical aspects of small waves remain poorly understood. Among them:

- Influence of the vorticity?
- Influence of the bottom (flat/periodic?)
- Influence of the air? (e.g. formation of waves, see **Bühler-Shatah-Walsh-Zeng**)

It would be great to be able to perturb around more complicated equilibriums:

- Multi-solitons

Model limitations

Most dynamical aspects of small waves remain poorly understood. Among them:

- Influence of the vorticity?
- Influence of the bottom (flat/periodic?)
- Influence of the air? (e.g. formation of waves, see **Bühler-Shatah-Walsh-Zeng**)

It would be great to be able to perturb around more complicated equilibriums:

- Multi-solitons
- Stability close to solitons (e.g. Instability result by **Rousset-Tzvetkov**)

Water waves and semilinear dispersive equations

Many dispersive equations appear as some limit from the WW:

- KdV in the some form of Shallow-water regime; also KP-I/II.

Water waves and semilinear dispersive equations

Many dispersive equations appear as some limit from the WW:

- KdV in the some form of Shallow-water regime; also KP-I/II.
- NLS for some modulations.

Water waves and semilinear dispersive equations

Many dispersive equations appear as some limit from the WW:

- KdV in the some form of Shallow-water regime; also KP-I/II.
- NLS for some modulations.
- Benjamin-Ono for internal waves.

Water waves and semilinear dispersive equations

Many dispersive equations appear as some limit from the WW:

- KdV in the some form of Shallow-water regime; also KP-I/II.
- NLS for some modulations.
- Benjamin-Ono for internal waves.

Water waves and semilinear dispersive equations

Many dispersive equations appear as some limit from the WW:

- KdV in the some form of Shallow-water regime; also KP-I/II.
- NLS for some modulations.
- Benjamin-Ono for internal waves.

See **Schneider-Wayne**, **Craig**, **Bona-Colin-Lannes**, **Alvarez-Samaniego-Lannes**, **Totz-Wu**.

Water waves and semilinear dispersive equations

Many dispersive equations appear as some limit from the WW:

- KdV in the some form of Shallow-water regime; also KP-I/II.
- NLS for some modulations.
- Benjamin-Ono for internal waves.

See **Schneider-Wayne**, **Craig**, **Bona-Colin-Lannes**, **Alvarez-Samaniego-Lannes**, **Totz-Wu**.

Semilinear equations better understood than WW equations. It would be great to understand what can be inferred from properties of these flows to properties of solutions to the WW problem (e.g. control of 1D NLS helps in scattering for gKdV, see **Killip-Kwon-Shao-Visan**).

Steady waves

The best understood setup is the case of steady or standing waves, with many contributions, even for large data/vorticity/stratifications.
Amick, Beale, Toland, Ioss, Plotnikov, Varvaruca, Constantine, Strauss, Wahlen, Hur, Walsh, Wheeler, Craig, Groves, Kirchgässner, Alazard-Métivier.

Global existence for 3D GCWW

GWP for small localized smooth perturbations of a flat $2D$ interface at rest, over an infinite bottom subject to gravity and surface tension.

GWP for small gravity-capillary waves [**DIPP**]

There exists a norm (finite on $\mathcal{S}$) and $\varepsilon > 0$ such that if (h, ϕ) solve the water-wave problem in ZCS formulation with

$$\|(h(0), \phi(0))\| \leq \varepsilon,$$

then (h, ϕ) can be extended globally and scatters in L^2,

$$\|U(t)\|_{L^\infty} \lesssim (1 + |t|)^{-\frac{5}{6}+}$$

and we have precise information on $U = \sqrt{g - \sigma\Delta}h + i|\nabla|^{\frac{1}{2}}\phi$

Previous works on WW

Among many previous works:

- Local well-posedness: **Nalimov**, **Yoshihara**, **Kano-Nishida**, **Beale-Hou-Lowengrub**, **Craig**, **Iguchi**, **Ogawa-Tani**, Wu, **Ambrose-Masmoudi**, **Christodoulou-Lindblad**, **Lannes**, **Lindblad**, **Coutand-Shkoller**, **Zhang-Zhang**, **Shatah-Zeng**, **Beyer-Gunther**, **Christianson-Hur-Staffilani**, Alazard-Burq-Zuily, **de Poyferre-NGuyen**.

Previous works on WW

Among many previous works:

- Global well-posedness in 2D (1d interface):
  - **Wu**, **Hunter-Ifrim-Tataru** (gravity, almost global), **Alazard-Delort**, **Ionescu-Pusateri**, **Ifrim-Tataru**, **X. Wang** (gravity)

Previous works on WW

Among many previous works:

- Global well-posedness in 2D (1d interface):
  - **Wu**, **Hunter-Ifrim-Tataru** (gravity, almost global), **Alazard-Delort**, **Ionescu-Pusateri**, **Ifrim-Tataru**, **X. Wang** (gravity)
  - **Ifrim-Tataru**, **Ionescu-Pusateri** (surface tension).

Previous works on WW

Among many previous works:

- Global well-posedness in 2D (1d interface):
  - **Wu**, **Hunter-Ifrim-Tataru** (gravity, almost global), **Alazard-Delort**, **Ionescu-Pusateri**, **Ifrim-Tataru**, **X. Wang** (gravity)
  - **Ifrim-Tataru**, **Ionescu-Pusateri** (surface tension).
- Global well-posedness in 3D (2d interface):

Previous works on WW

Among many previous works:

- Global well-posedness in 2D (1d interface):
  - **Wu**, **Hunter-Ifrim-Tataru** (gravity, almost global), **Alazard-Delort**, **Ionescu-Pusateri**, **Ifrim-Tataru**, **X. Wang** (gravity)
  - **Ifrim-Tataru**, **Ionescu-Pusateri** (surface tension).
- Global well-posedness in 3D (2d interface):
  - **Germain-Masmoudi-Shatah**, **Wu** (gravity), **X. Wang** (finite bottom),

Previous works on WW

Among many previous works:

- Global well-posedness in 2D (1d interface):
  - **Wu**, **Hunter-Ifrim-Tataru** (gravity, almost global), **Alazard-Delort**, **Ionescu-Pusateri**, **Ifrim-Tataru**, **X. Wang** (gravity)
  - **Ifrim-Tataru**, **Ionescu-Pusateri** (surface tension).
- Global well-posedness in 3D (2d interface):
  - **Germain-Masmoudi-Shatah**, **Wu** (gravity), **X. Wang** (finite bottom),
  - **Germain-Masmoudi-Shatah** (surface tension).

Specificity of the gravity-capillary problem

The case of gravity-capillary WW presents serious new difficulties:

- Slower linear decay ($t^{-\frac{5}{6}} \rightarrow$ *nonintegrable*)

Specificity of the gravity-capillary problem

The case of gravity-capillary WW presents serious new difficulties:

- Slower linear decay ($t^{-\frac{5}{6}} \rightarrow$ *nonintegrable*)
- Quadratic resonances $\rightarrow$ no normal form

Specificity of the gravity-capillary problem

The case of gravity-capillary WW presents serious new difficulties:

- Slower linear decay ($t^{-\frac{5}{6}} \rightarrow$ *nonintegrable*)
- Quadratic resonances $\rightarrow$ no normal form
- No scaling invariance

Specificity of the gravity-capillary problem

The case of gravity-capillary WW presents serious new difficulties:

- Slower linear decay ($t^{-\frac{5}{6}} \to$ *nonintegrable*)
- Quadratic resonances $\to$ no normal form
- No scaling invariance
- Presence of space-time resonances $\to$ delicate semilinear analysis

Dispersion relation

Linearize at equilibrium:

$$\left(\partial_t + i\sqrt{|\nabla|(g - \sigma\Delta)}\right) U = 0.$$

Solve by Fourier transform:

$$U(t) = e^{-it\Lambda} U(0),$$

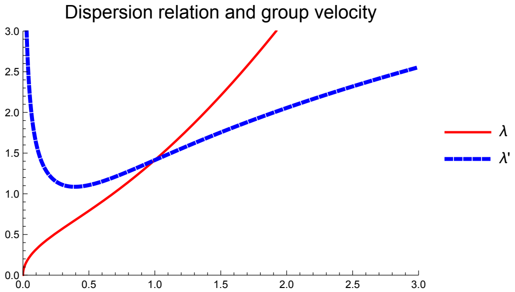
Need to understand dispersive properties of free solutions: Dispersion relation

$$\Lambda(\xi) = \lambda(|\xi|), \qquad \lambda(r) = \sqrt{gr + \sigma r^3}.$$

Dispersion relation

Solve by Fourier transform:

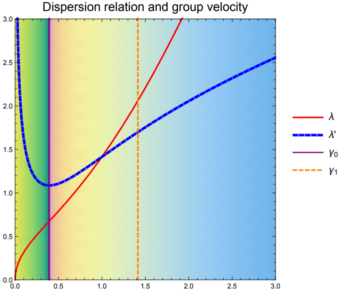
$$U(t) = e^{-it\Lambda}U(0), \qquad \lambda(r) = \sqrt{g r + \sigma r^3}$$



Dispersion relation

Solve by Fourier transform:

$$U(t) = e^{-it\Lambda}U(0), \qquad \lambda(r) = \sqrt{gr + \sigma r^3}$$



In water, $\gamma_0 \sim 58\text{m}^{-1}$, $2\pi/\gamma_0 \simeq 1.7\text{cm}$.

Slower linear decay

Inflexion point in the dispersion relation (γ_0) $\rightarrow$ slower decay, Van Der Corput:

$$\|e^{it\Lambda}P_N f\|_{L^\infty} \lesssim \min\{N^{\frac{3}{2}}, N^{\frac{1}{2}}\} t^{-1+\frac{1}{6}} \|P_N f\|_{L^1}.$$

Loosing "almost integrable decay" leads to many complications.

Strategy

We proceed in two main steps

1. Energy estimates: allows to control high regularity norms of the solution assuming good dispersive behavior of low frequencies,

Strategy

We proceed in two main steps

1. Energy estimates: allows to control high regularity norms of the solution assuming good dispersive behavior of low frequencies,
2. Dispersive analysis: allows to control the dispersive behavior of the solution assuming good energy estimates.

Zakharov-Craig-Sulem equation

Restrict to the boundary (graph over equilibrium $h \equiv 0$):

$$(h, \phi), \qquad \phi(x) = \Phi(x, h(x)), \qquad v = \nabla\Phi, \quad \Delta\Phi = 0,$$
$$G(h)\phi = \sqrt{1 + |\nabla h|^2}\mathbf{n} \cdot \nabla\Phi_{|y=h(x)}$$

Equations become

$$\partial_t h = G(h)\phi,$$
$$\partial_t \phi = -gh - \sigma \mathrm{div}\left[\frac{\nabla h}{[1 + |\nabla h|^2]^{\frac{3}{2}}}\right] - \frac{1}{2}|\nabla\phi|^2 + \frac{(G(h)\phi + \nabla h \cdot \nabla\phi)^2}{1 + |\nabla h|^2}$$

Hamiltonian flow

This is the *Hamiltonian flow* associated to the usual symplectic structure and to the physical energy:

$$\mathcal{H}(h,\phi)=\frac{1}{2}\int_{\mathbb{R}^2}\left\{\phi\cdot G(h)\phi+gh^2+2\sigma\left[\sqrt{1+|\nabla h|^2}-1\right]\right\}dx$$

Outline

Semilinear approach

Assume smoothness of solution (EE) $\rightarrow$ Taylor expansion:

$$(\partial_t + i\Lambda)\, U = Q(U, U) + C(U, U, U) + h.o.t,$$
$$\Lambda = \sqrt{|\nabla|(g - \sigma\Delta)}, \qquad U = \sqrt{g - \sigma\Delta}h + i|\nabla|^{\frac{1}{2}}\phi.$$

Want: decay of solutions $\rightarrow$ need to to exploit dispersive effects. Conjugating by the linear flow:

$$U(t) = e^{-it\Lambda}u(t),$$

then u evolves nonlinearly

$$\partial_t u = \text{quadratic} + h.o.t., \qquad \|U\|_{L^\infty} \lesssim t^{-\frac{5}{6}}\|u\|_{W^{2,1}}$$

New unknown: u.

Vector fields

Want: use stationary-phase arguments to obtain decay:

$$U(x,t) = \int_{\mathbb{R}^2} e^{i[t\Lambda(\xi)+\langle x,\xi\rangle]}\widehat{u}(\xi,t)d\xi$$

Need: smoothness of $\widehat{u}$.

Vector fields

Want: use stationary phase arguments to obtain decay:

$$U(x,t) = \int_{\mathbb{R}^2} e^{i[t\Lambda(\xi)+\langle x,\xi\rangle]}\widehat{u}(\xi,t)d\xi$$

Need: smoothness of $\widehat{u}$.

Vector-fields (**Klainerman**): look for $\mathcal{X}$ such that

- $\widehat{\mathcal{X}}$ also vector field,
- $\widehat{\mathcal{X}}$ commutes to first order with linearized operator.

Then: similar properties to $\widehat{\mathcal{X}_e} = \nabla \to$ energy estimates.
Informally: need 2 such vector fields.

Vector fields

Want: use stationary phase arguments to obtain decay:

$$U(x,t) = \int_{\mathbb{R}^2} e^{i[t\Lambda(\xi)+\langle x,\xi\rangle]}\widehat{u}(\xi,t)d\xi$$

Need: smoothness of $\widehat{u}$.

Vector-fields

- Isotropic problem: rotational vector field: $\mathcal{X} = \Omega$, $\widehat{\mathcal{X}} = \Omega$,
- ???

  Remark: in case of only gravity or surface tension, scaling invariance: $S = x\cdot\nabla + ct\partial_t + c'$.

Smoothness in Fourier space

To compensate for the missing vector field, we will try to obtain full smoothness (cf **Germain-Masmoudi-Shatah**, **Gustafson-Nakanishi-Tsai**):

$$\|\widehat{u}(t)\|_{H^s} \simeq \|\langle x\rangle^s u(t)\|_{L^2}.$$

In general, need $s = 1 = (d/2)$. Here 1 vector field $\to s > 1/2$.

Smoothness in Fourier space

To compensate for the missing vector field, we will try to obtain full smoothness (cf **Germain-Masmoudi-Shatah**, **Gustafson-Nakanishi-Tsai**):

$$\|\widehat{u}(t)\|_{H^s} \simeq \|\langle x\rangle^s u(t)\|_{L^2}.$$

In general, need $s = 1 = (d/2)$. Here 1 vector field $\to s > 1/2$. First attempt for a decay norm:

$$\|u(t)\|_{B_1} := \|\langle x\rangle^{1-\delta} u(t)\|_{L^2}$$

Smoothness in Fourier space

To compensate for the missing vector field, we will try to obtain full smoothness (cf **Germain-Masmoudi-Shatah**, **Gustafson-Nakanishi-Tsai**):

$$\|\widehat{u}(t)\|_{H^s} \simeq \|\langle x\rangle^s u(t)\|_{L^2}.$$

In general, need $s = 1 = (d/2)$. Here 1 vector field $\to s > 1/2$. First attempt for a decay norm:

$$\|u(t)\|_{B_1} := \|\langle x\rangle^{1-\delta} u(t)\|_{L^2}$$

Insufficient!

A nonlinear norm

Previous discussion only based solely on linear considerations.

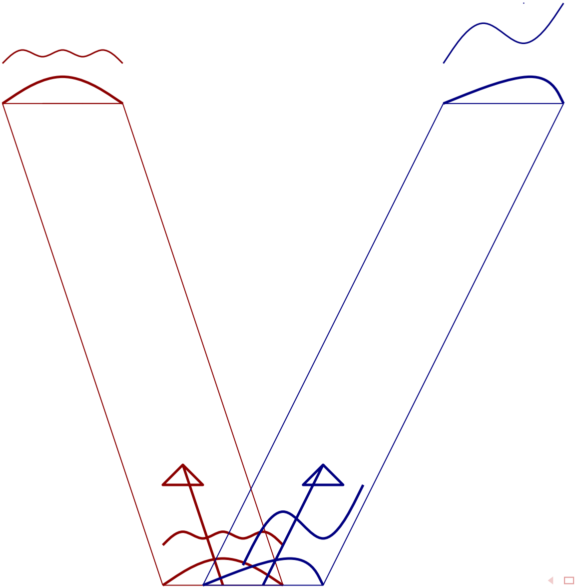
A nonlinear norm

Previous discussion only based solely on linear considerations.
Special nonlinear interactions prevents propagation of smooth norms.

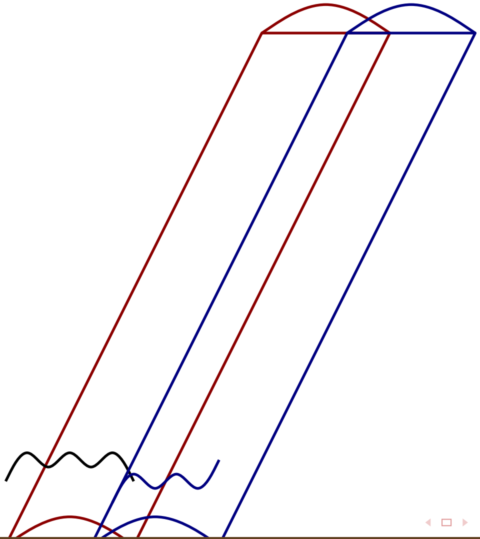
A nonlinear norm

Previous discussion only based solely on linear considerations.
Special nonlinear interactions prevents propagation of smooth norms.
$\rightarrow$ Need to modify the norm (Norm dependent on nonlinearity).

Quadratic space-time resonances



Space-resonant/Coherent interaction: Same velocity



Space-time resonant interactions

Interactions of waves at frequencies ξ_1 and ξ_2 such that

$$\begin{aligned} \nabla\Lambda(\xi_1) = \nabla\Lambda(\xi_2) \quad &\Leftrightarrow \nabla_\eta\Phi = 0, \\ \Lambda(\xi_1) + \Lambda(\xi_2) = \Lambda(\xi_1 + \xi_2) \quad &\Leftrightarrow \Phi = 0, \end{aligned}$$

cannot in general be avoided: $(D+1)$ equations in $2D$ dimensions.

Space-time resonant interactions

Interactions of waves at frequencies ξ_1 and ξ_2 such that

$$\begin{aligned} \nabla\Lambda(\xi_1) = \nabla\Lambda(\xi_2) \quad &\Leftrightarrow \nabla_\eta\Phi = 0, \\ \Lambda(\xi_1) + \Lambda(\xi_2) = \Lambda(\xi_1+\xi_2) \quad &\Leftrightarrow \Phi = 0, \end{aligned}$$

cannot in general be avoided: $(D+1)$ equations in $2D$ dimensions.
Create space-time resonances (terminology of **Germain-Masmoudi-Shatah**).

Space-time resonant interactions

Interactions of waves at frequencies ξ_1 and ξ_2 such that

$$\begin{aligned} \nabla\Lambda(\xi_1) &= \nabla\Lambda(\xi_2) \quad &\Leftrightarrow \nabla_\eta\Phi = 0, \\ \Lambda(\xi_1)+\Lambda(\xi_2) &= \Lambda(\xi_1+\xi_2) \quad &\Leftrightarrow \Phi = 0, \end{aligned}$$

cannot in general be avoided: $(D+1)$ equations in $2D$ dimensions.
Create space-time resonances (terminology of **Germain-Masmoudi-Shatah**).
Presence of space-time resonances makes analysis more complicated. **Bernicot-Germain** studied the first iterate:

- no correction to optimal decay in $3D$: $1/t^{3/2}$,

Space-time resonant interactions

Interactions of waves at frequencies ξ_1 and ξ_2 such that

$$\begin{aligned} \nabla\Lambda(\xi_1) = \nabla\Lambda(\xi_2) &\quad \Leftrightarrow \nabla_\eta \Phi = 0, \\ \Lambda(\xi_1) + \Lambda(\xi_2) = \Lambda(\xi_1 + \xi_2) &\quad \Leftrightarrow \Phi = 0, \end{aligned}$$

cannot in general be avoided: $(D+1)$ equations in $2D$ dimensions.
Create space-time resonances (terminology of **Germain-Masmoudi-Shatah**).
Presence of space-time resonances makes analysis more complicated. **Bernicot-Germain** studied the first iterate:

- no correction to optimal decay in $3D$: $1/t^{3/2}$,
- best decay in $2D$: $\log(t)/t$

Space-time resonant interactions

Interactions of waves at frequencies ξ_1 and ξ_2 such that

$$\begin{aligned} \nabla\Lambda(\xi_1) = \nabla\Lambda(\xi_2) \quad &\Leftrightarrow \nabla_\eta\Phi = 0, \\ \Lambda(\xi_1) + \Lambda(\xi_2) = \Lambda(\xi_1+\xi_2) \quad &\Leftrightarrow \Phi = 0, \end{aligned}$$

cannot in general be avoided: $(D+1)$ equations in $2D$ dimensions.
Create space-time resonances (terminology of **Germain-Masmoudi-Shatah**).
Presence of space-time resonances makes analysis more complicated. **Bernicot-Germain** studied the first iterate:

- no correction to optimal decay in $3D$: $1/t^{3/2}$,
- best decay in $2D$: $\log(t)/t$
- best decay in $1D$: $t^{\frac{1}{4}}/t^{1/2}$.

Space-time resonant interactions

Interactions of waves at frequencies ξ_1 and ξ_2 such that

$$\nabla\Lambda(\xi_1) = \nabla\Lambda(\xi_2) \quad \Leftrightarrow \nabla_\eta\Phi = 0,$$
$$\Lambda(\xi_1) + \Lambda(\xi_2) = \Lambda(\xi_1+\xi_2) \quad \Leftrightarrow \Phi = 0,$$

cannot in general be avoided: $(D+1)$ equations in $2D$ dimensions.
Create space-time resonances (terminology of **Germain-Masmoudi-Shatah**).
Presence of space-time resonances makes analysis more complicated. **Bernicot-Germain** studied the first iterate:

- no correction to optimal decay in $3D$: $1/t^{3/2}$,
- best decay in $2D$: $\log(t)/t$
- best decay in $1D$: $t^{\frac{1}{4}}/t^{1/2}$.

Space-time resonant interactions

Interactions of waves at frequencies ξ_1 and ξ_2 such that

$$\begin{aligned} \nabla\Lambda(\xi_1) &= \nabla\Lambda(\xi_2) \quad &\Leftrightarrow \nabla_\eta\Phi = 0, \\ \Lambda(\xi_1) + \Lambda(\xi_2) &= \Lambda(\xi_1 + \xi_2) \quad &\Leftrightarrow \Phi = 0, \end{aligned}$$

cannot in general be avoided: $(D+1)$ equations in $2D$ dimensions.
Create space-time resonances (terminology of **Germain-Masmoudi-Shatah**).
Presence of space-time resonances makes analysis more complicated. **Bernicot-Germain** studied the first iterate:

- no correction to optimal decay in $3D$: $1/t^{3/2}$,
- best decay in $2D$: $\log(t)/t$
- best decay in $1D$: $t^{\frac{1}{4}}/t^{1/2}$.

Genuinely nonlinear effect!

STR are generic $\rightarrow$ appear in many dispersive systems. E.g. in Euler-Maxwell systems.
Germain: isolates a nice model setting: systems of Klein-Gordon equations (cf **Germain-Masmoudi**).

STR are generic → appear in many dispersive systems. E.g. in Euler-Maxwell systems.
Germain: isolates a nice model setting: systems of Klein-Gordon equations (cf **Germain-Masmoudi**).

Robust study in non-degenerate case: **Ionescu-P.**, see also **Yu Deng**'s thesis for some degenerate cases.

STR are generic → appear in many dispersive systems. E.g. in Euler-Maxwell systems.
Germain: isolates a nice model setting: systems of Klein-Gordon equations (cf **Germain-Masmoudi**).

Robust study in non-degenerate case: **Ionescu-P.**, see also **Yu Deng**'s thesis for some degenerate cases.

Led to global solutions for the 2-fluid Euler-Maxwell system in $3D$ **Guo-Ionescu-P.** and electron system in $2D$ **Deng-Ionescu-P.**.

Duhamel formula

Introducing quadratic interactions:

$$(\partial_t + i\Lambda)\, U = Q_1[U, U] + Q_2[U, \overline{U}] + Q_3[\overline{U}, \overline{U}] + h.o.t.$$

Duhamel formula

Introducing quadratic interactions:

$$(\partial_t + i\Lambda)\, U = Q_1[U, U] + Q_2[U, \overline{U}] + Q_3[\overline{U}, \overline{U}] + h.o.t.$$

Duhamel formula for u:

$$\widehat{u}(\xi, t) = \widehat{u}(\xi, 0) - i \int_0^t \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)}\mathfrak{m}(\xi, \eta)\widehat{u}(\xi - \eta, s)\widehat{u}(\eta, s)d\eta ds,$$

$$\Phi(\xi, \eta) = \Lambda(\xi) - \Lambda(\xi - \eta) - \Lambda(\eta),$$

Duhamel formula

Introducing quadratic interactions:

$$(\partial_t + i\Lambda)\, U = Q_1[U, U] + Q_2[U, \overline{U}] + Q_3[\overline{U}, \overline{U}] + h.o.t.$$

Duhamel formula for u:

$$\widehat{u}(\xi, t) = \widehat{u}(\xi, 0) - i \int_0^t \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} \mathfrak{m}(\xi, \eta) \widehat{u}(\xi - \eta, s) \widehat{u}(\eta, s) d\eta ds,$$

$$\Phi(\xi, \eta) = \Lambda(\xi) - \Lambda(\xi - \eta) - \Lambda(\eta),$$

Coherence-resonance information contained in the stationary points of the phase $s\Phi$ (*space-time resonance method* **Germain-Masmoudi-Shatah**).

Non degenerate STR

We call an interaction nondegenerate if

$$\Phi = 0 \ \&\ \nabla_\eta \Phi = 0 \quad \Rightarrow \quad \det \nabla^2_{\eta\eta} \Phi \neq 0$$

generically satisfied for isotropic problems (true for GCWW).

Non degenerate STR

We call an interaction nondegenerate if

$$\Phi = 0 \ \& \ \nabla_\eta \Phi = 0 \quad \Rightarrow \quad \det \nabla^2_{\eta\eta} \Phi \neq 0$$

generically satisfied for isotropic problems (true for GCWW).

Effect of coherent interaction:

$$\partial_t \widehat{u}(\xi) = \frac{1}{t} e^{it\Psi(\xi)} g(\xi) + R(\xi), \qquad g \in C^\infty, \quad \|R\|_{L^2} \lesssim (1+|t|)^{-\frac{3}{2}},$$

$$\Psi(\xi) = \Phi(\xi,\eta) \quad \text{when} \quad \nabla_\eta \Phi(\xi,\eta) = 0,$$

Non degenerate STR

We call an interaction nondegenerate if

$$\Phi = 0 \ \& \ \nabla_\eta \Phi = 0 \quad \Rightarrow \quad \det \nabla^2_{\eta\eta}\Phi \neq 0$$

generically satisfied for isotropic problems (true for GCWW).

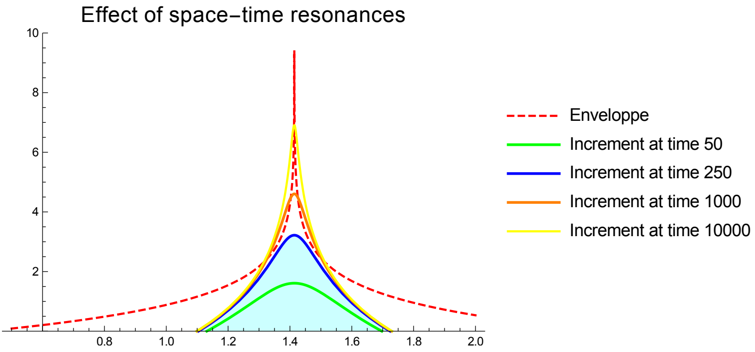
Effect of coherent interaction:

$$\partial_t \widehat{u}(\xi) = \frac{1}{t} e^{it\Psi(\xi)} g(\xi) + R(\xi), \qquad g \in C^\infty, \quad \|R\|_{L^2} \lesssim (1+|t|)^{-\frac{3}{2}},$$

$$\Psi(\xi) = \Phi(\xi,\eta) \quad \text{when} \quad \nabla_\eta \Phi(\xi,\eta) = 0,$$

First term does not behave as a linear wave
$\rightarrow$ treated differently!

Build up due to space-time resonance



Norms

We will bootstrap control of the following norms:

$$\sup_{0\le t\le T}\left\{\|U(t)\|_{H^{N_0}}+\|\Omega^{N_2}U(t)\|_{L^2}+\|u(t)\|_Z\right\}\le\varepsilon,$$

$$\|f\|_Z\lesssim\sup_{0\le a\le N_2/2}\|\Omega^a f\|_{\widetilde{Z}},$$

$$\|f\|_{\widetilde{Z}}=\sup_{N\cdot X\ge 1}(1+N^{30})\|Q_{X,N}f\|_{B^1_{X,N}+B^2_{X,N}},$$

$$\|f\|_{B^1_{X,N}}=X^{1-9\delta}\|f\|_{L^2},$$

$$\|f\|_{B^2_{X,N}}\sim(N^{-10}+N^{10})X^{1-\delta}\|\Psi(\xi)\widehat{f}(\xi)\|_{L^\infty},\qquad \Psi(\xi)\simeq|\xi|-\sqrt{2}$$

where $Q_{X,N}$ localizes at frequency $|\xi|\simeq N$ and at position $|x|\simeq X$.

Norms

We will bootstrap control of the following norms:

$$\sup_{0\leq t\leq T}\left\{\|U(t)\|_{H^{N_0}}+\|\Omega^{N_2}U(t)\|_{L^2}+\|u(t)\|_Z\right\}\leq\varepsilon,$$

$$\|f\|_Z\lesssim\sup_{0\leq a\leq N_2/2}\|\Omega^a f\|_{\widetilde{Z}},$$

$$\|f\|_{\widetilde{Z}}=\sup_{N\cdot X\geq 1}(1+N^{30})\|Q_{X,N}f\|_{B^1_{X,N}+B^2_{X,N}},$$

$$\|f\|_{B^1_{X,N}}=X^{1-9\delta}\|f\|_{L^2},$$

$$\|f\|_{B^2_{X,N}}\sim(N^{-10}+N^{10})X^{1-\delta}\|\Psi(\xi)\widehat{f}(\xi)\|_{L^\infty},\qquad \Psi(\xi)\simeq|\xi|-\sqrt{2}$$

where $Q_{X,N}$ localizes at frequency $|\xi|\simeq N$ and at position $|x|\simeq X$.

Norms

We will bootstrap control of the following norms:

$$\sup_{0\leq t\leq T}\left\{\|U(t)\|_{H^{N_0}}+\|\Omega^{N_2}U(t)\|_{L^2}+\|u(t)\|_Z\right\}\leq\varepsilon,$$

$$\|f\|_Z\lesssim\sup_{0\leq a\leq N_2/2}\|\Omega^a f\|_{\widetilde{Z}},$$

$$\|f\|_{\widetilde{Z}}=\sup_{N\cdot X\geq 1}(1+N^{30})\|Q_{X,N}f\|_{B^1_{X,N}+B^2_{X,N}},$$

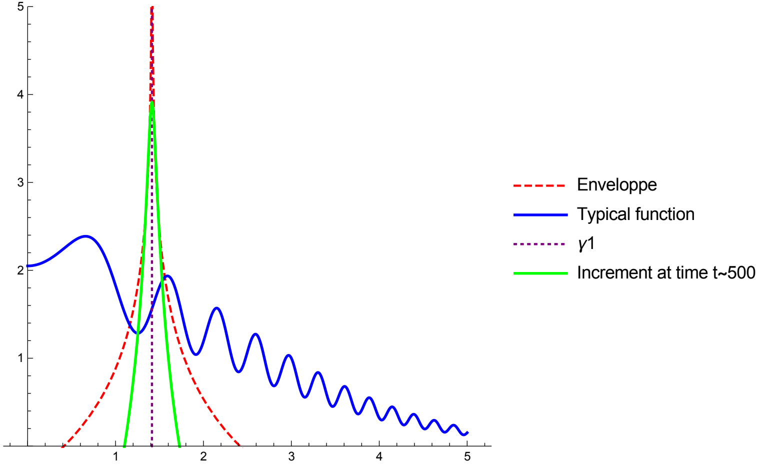
$$\|f\|_{B^1_{X,N}}=X^{1-9\delta}\|f\|_{L^2},$$

$$\|f\|_{B^2_{X,N}}\sim(N^{-10}+N^{10})X^{1-\delta}\|\Psi(\xi)\widehat{f}(\xi)\|_{L^\infty},\qquad \Psi(\xi)\simeq|\xi|-\sqrt{2}$$

where $Q_{X,N}$ localizes at frequency $|\xi|\simeq N$ and at position $|x|\simeq X$.

$\|U(t)\|_{L^\infty}\leq t^{-\frac{5}{6}+}\|u\|_Z$

A typical function



Main properties

The following generic assumptions are verified for our problem:

- **STR** are separated: if (ξ, η) is STR, then (η, χ) is not STR for any χ. No **STR** feeds into another **STR**.

Main properties

The following generic assumptions are verified for our problem:

- **STR** are separated: if (ξ, η) is STR, then (η, χ) is not STR for any χ. No **STR** feeds into another **STR**.
- No nontrivial iterated resonances: if (ξ, η) is a **STR** and (χ, ξ) is resonant which is coherent with the new wave, then they have the same speed:

Main properties

The following generic assumptions are verified for our problem:

- **STR** are separated: if (ξ, η) is STR, then (η, χ) is not STR for any χ. No **STR** feeds into another **STR**.
- No nontrivial iterated resonances: if (ξ, η) is a **STR** and (χ, ξ) is resonant which is coherent with the new wave, then they have the same speed:

Main properties

The following generic assumptions are verified for our problem:

- **STR** are separated: if (ξ, η) is STR, then (η, χ) is not STR for any χ. No **STR** feeds into another **STR**.
- No nontrivial iterated resonances: if (ξ, η) is a **STR** and (χ, ξ) is resonant which is coherent with the new wave, then they have the same speed:

$$\Phi_1 + \Psi_2 = \Lambda_\sigma(\xi) - \Lambda_\mu(\xi - \eta) - \Lambda_\nu(\eta) + [\Lambda_\nu(\eta) + \Lambda_\mu(\eta - \theta) - \Lambda_\sigma(\theta$$
$$\nabla_\eta \Phi_2 = \nabla\Lambda_\mu(\eta - \theta) + \nabla\Lambda_\sigma(\theta) = 0$$

then $\xi = \theta$.

Figure: Separation of resonances (generic)

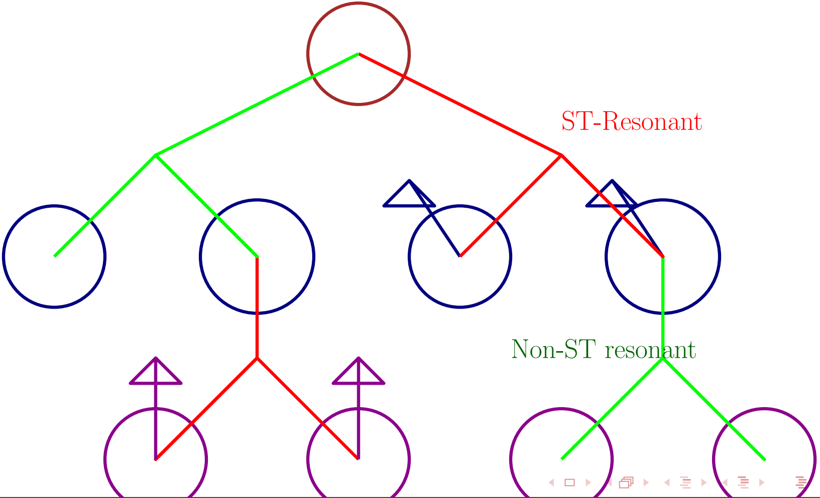
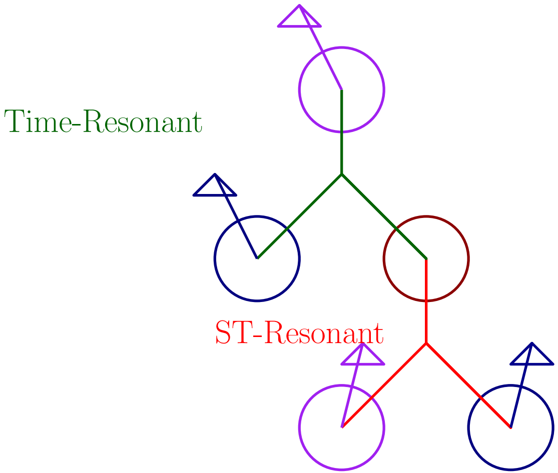


Figure: Local iterated resonances (generic)



Decompose the inputs all the way to the uncertainty principle

$$u = \sum_{N \cdot X \geq 1} Q_{X,N} u, \qquad Q_{X,N} \simeq \mathbf{1}_{|x| \simeq X} \mathbf{1}_{|\xi| \simeq N}$$

and we plug in the Duhamel formula

$$\widehat{u}(\xi, t) = \widehat{u}(\xi, 0) - i \sum_{X_1 \cdot N_1 \geq 1} \sum_{X_2 \cdot N_2 \geq 1} \int_0^t e^{it\Phi(\xi,\eta)} \widehat{u}_{X_1,N_1}(\xi - \eta) \widehat{u}_{X_2,N_2}(\eta) d\eta,$$

and we estimate $Q_{X,N} u$.

- By energy estimates, we may assume that $N, N_1, N_2 \lesssim 1$.

This allows to evacuate most of the easy cases.

- By energy estimates, we may assume that $N, N_1, N_2 \lesssim 1$.
- By finite speed of propagation, we may assume that $X, X_1, X_2 \leq T$ and then T is the largest parameter.

This allows to evacuate most of the easy cases.

“Local” slow frequency

At the inflexion point γ_0, particularly slow decay $\to$ rely more on normal form transformation.

"Local" slow frequency

At the inflexion point γ_0, particularly slow decay $\rightarrow$ rely more on normal form transformation.

Key: γ_0 has the slowest group velocity $\rightarrow$ interacts mostly with itself $\rightarrow$ local ($\sim |u|^2 u$).

Main cases

Separation assumption: a ST resonance created *only* by "strong" inputs (already studied in $3D$).

Hardest case: one input is Schwartz and one is very delocalized: $X_1 \sim 0$, $X_2 \simeq T$. No space-resonances/coherence analysis. Need to reiterate the analysis (cf EP/e 2D).