On global solutions of water wave models

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The "division" problem

Consider a generic evolution problem of the type

$$\partial_t u + i\Lambda u = \mathcal{N}(u, D_x u)$$

where Λ is real and $\mathcal{N}$ is a quadratic nonlinearity. At first iteration

$$u(t) = e^{-it\Lambda}\phi.$$

At second iteration, assuming $\mathcal{N} = \partial_1(u^2)$,

$$\widehat{u}(\xi, t) = e^{-it\Lambda(\xi)}\widehat{\phi}(\xi)$$
$$+ Ce^{-it\Lambda(\xi)} \int \widehat{\phi}(\xi - \eta)\widehat{\phi}(\eta) i\xi_1 \frac{1 - e^{it[\Lambda(\xi)-\Lambda(\eta)-\Lambda(\xi-\eta)]}}{\Lambda(\xi) - \Lambda(\eta) - \Lambda(\xi - \eta)}\, d\eta.$$

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In semilinear problems one can iterate using $X^{s,b}$ spaces (Bourgain, Kenig–Ponce–Vega, Klainerman–Machedon). The iteration method completely fails in quasilinear problems due to the unavoidable loss of derivative.

In quasilinear problems, the classical methods are

- energy and vector-field methods (Klainerman, Christodoulou);
- the normal form method (Shatah).

There are two possible situations: low regularity + short time or high regularity + long time. We focus on the second case.

In many interesting quasilinear evolutions it is not known how to construct even one dynamically nontrivial global solution.

We assume that we start with "nice" and "small" data (smooth and localized) and would like to understand the long-term evolution problem. The exact smoothness assumption is not important, but the "momentum" conditions (assumption on the 0 frequency) are important.

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In quasilinear problems, understanding the division problem is necessary when:

- The solution has strictly less than $1/t$ pointwise decay;
- There is a full set (codimension 1) of time resonances and no matching "null structure".

The main point is that the phases corresponding to bilinear interactions satisfy the following **restricted nondegeneracy condition**: if

$$\Phi(\xi,\eta) := \pm\Lambda(\xi) \pm \Lambda(\eta) \pm \Lambda(\xi-\eta)$$

and

$$\Upsilon(\xi,\eta) := \nabla^2_{\xi,\eta}\Phi(\xi,\eta)\left[\nabla^{\perp}_{\xi}\Phi(\xi,\eta), \nabla^{\perp}_{\eta}\Phi(\xi,\eta)\right],$$

then $\Upsilon(\xi,\eta) \neq 0$ at (almost all) points on the time-resonant set $\Phi(\xi,\eta) = 0$.

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We have three results in this direction:

- I.–Pusateri: the capillary water waves in 2d with no momentum conditions on the Hamiltonian variables (independent result of Ifrim–Tataru in the case of data satisfying one momentum condition on the Hamiltonian variable);

- Deng–I.–Pausader: the Euler–Maxwell one-fluid system in 2d.

- Deng–I.–Pausader–Pusateri: the gravity-capillary irrotational water waves in 3d (2d interface). Schematically,

$$\partial_t u + i\Lambda u = \mathcal{N}(u, D_x u), \qquad \Lambda(\xi) = \sqrt{|\xi| + |\xi|^3}.$$

The main theorem

We consider the free boundary incompressible Euler equations

$$v_t + v \cdot \nabla v = -\nabla p - g e_n, \qquad \nabla \cdot v = 0, \qquad x \in \Omega_t,$$

where g is the gravitational constant. The free surface $S_t = \{(x, h(x,t))\}$ moves with the velocity, according to the kinematic boundary condition:

$$\partial_t + v \cdot \nabla \text{ is tangent to } \bigcup_t S_t \subset \mathbf{R}^{n+1}.$$

In the presence of surface tension the pressure on the interface is given by

$$p(x,t) = \sigma \kappa(x,t), \quad x \in S_t\,,$$

where κ is the mean-curvature of S_t and $\sigma > 0$.

In the irrotational case $\mathrm{curl}\, v = 0$, let Φ denote the velocity potential, $v = \nabla\Phi$, and let $\phi(x,t) = \Phi(x, h(x,t), t)$ denote its trace on the interface.

Main Theorem. (Deng, I., Pausader, Pusateri) If $g > 0$, $\sigma > 0$, and

$$\|(h_0, \phi_0)\|_{\text{Suitable norm}} \leq \varepsilon_0 \ll 1$$

then there is a unique smooth global solution of the gravity-capillary water-wave system in 3d, with initial data (h_0, ϕ_0). The solution $(h, \phi)(t)$ decays in L^∞ at $t^{-5/6+}$ rate as $t \to \infty$.

Model equation:

$$(\partial_t + i\Lambda)U = iT_{v\cdot\zeta}U,$$

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The quasilinear I-method

The I-method of Colliander–Keel–Staffilani–Takaoka–Tao is a method to estimate the increment of energy:

- Start with an energy inequality of the form

$$\mathcal{E}_N(t) - \mathcal{E}_N(0) \leq \left| \int_0^t \int_{\mathbb{R}^d} D^N U \times D^N U \times DU \, dxdt \right|$$

- Transfer to the Fourier space,

$$\mathcal{E}_N(t) - \mathcal{E}_N(0) \leq \left| \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \widehat{D^N U}(\xi) \widehat{D^N U}(\eta) \widehat{DU}(-\xi-\eta) \, d\xi d\eta dt \right|$$

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- Use the equation, schematically,

$$(\partial_t + i\Lambda)U = D(U \times U).$$

Write $V = e^{it\Lambda}U$ (extract the free flow),

$$\partial_t V = e^{it\Lambda} D(e^{-it\Lambda}V \times e^{-it\Lambda}V).$$

- Rewrite the energy increment inequality in terms of the profile V

$$\mathcal{E}_N(t) - \mathcal{E}_N(0) \leq \Big| \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{it\Phi(\xi,\eta)} \widehat{D^N V}(\xi) \widehat{D^N V}(\eta) \widehat{DV}(-\xi-\eta) \, d\xi d\eta dt \Big|$$

Here

$$\Phi(\xi,\eta) = \pm\Lambda(\xi) \pm \Lambda(\eta) \pm \Lambda(-\xi-\eta).$$

The function Φ (typically) has a codimension 1 vanishing set. For example, in the gravity–capillary problem

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- Decompose the bulk term, $I = I_1 + I_2$,

$$I_1 := \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{it\Phi(\xi,\eta)} \varphi_{\le p}(\Phi(\xi,\eta)) \widehat{D^N V}(\xi)\widehat{D^N V}(\eta)\widehat{DV}(-\xi-\eta)\, d\xi d\eta dt,$$

$$I_2 := \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{it\Phi(\xi,\eta)} \varphi_{> p}(\Phi(\xi,\eta)) \widehat{D^N V}(\xi)\widehat{D^N V}(\eta)\widehat{DV}(-\xi-\eta)\, d\xi d\eta dt,$$

for a suitable choice of $p = p(m,k)$, $|\xi| \approx 2^k \gg 1$, $|t| \approx 2^m \gg 1$, $|\xi+\eta| \lesssim 1$.

- Estimate $|I_1|$ using an L^2 lemma. This is the critical gain of the method. It depends on the functions Φ satisfying the "restricted nondegeneracy condition".

- Estimate $|I_2|$ using integration by parts in time (Shatah's normal form method) and the equation for V

$$\partial_t V = e^{it\Lambda} D(e^{-it\Lambda} V \times e^{-it\Lambda} V).$$

One needs to use careful symmetrization to avoid the potential loss of derivative coming from the quasilinear nature of the equation. We identify a **strongly semilinear** structure in the bulk integrals, essentially a gain of one derivative in the region where $|\Phi| \lesssim 1$.

- Prove an identity of the form

$$|\mathcal{E}_N(t) - \mathcal{E}_N(0)| \lesssim \text{Cubic term with special structure} + \text{Quartic term}.$$

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Main L^2 lemma: Assume that $k, m \gg 1$,

$$p \geq -(1-\delta)m, \qquad p-k/2 \leq -(1/3+\delta)m, \qquad 2^{m-1} \leq |s| \leq 2^{m+1}.$$

Let T_p denote the operator defined by

$$T_p f(\xi) := \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)}\chi(2^{-p}\Phi(\xi,\eta))\chi_{\gamma_0}(\xi-\eta)\varphi_k(\eta)a(\xi,\eta)f(\eta)d\eta.$$

where

$$\Phi(\xi,\eta) = \Lambda(\xi) \pm \Lambda(\xi-\eta) - \Lambda(\eta),$$

Then

$$\|T_p\|_{L^2\to L^2} \lesssim 2^{\delta^2 m}[2^{-m/3+(p-k/2)} + 2^{3(p-k/2)/2}].$$

Depends on the fact that

$$\Upsilon(\xi,\eta) := \nabla^2_{\xi,\eta}\Phi(\xi,\eta)\left[\nabla^\perp_\xi\Phi(\xi,\eta), \nabla^\perp_\eta\Phi(\xi,\eta)\right] \neq 0,$$

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The kernel of the $T_p T_p^*$ operator is

$$K(x,\xi) := \int_{\mathbb{R}^2} e^{is\Theta(x,\xi,y)}\chi(2^{-p}\Phi(x,y))\chi(2^{-p}\Phi(\xi,y))a(x,\xi,y)dy,$$

$$\Theta(x,\xi,y) := \Phi(x,y) - \Phi(\xi,y) = \Lambda(x) - \Lambda(\xi) - \Lambda(x-y) + \Lambda(\xi-y).$$

where $|x|, |\xi|, |y| \approx 2^k$, $|x-y|, |\xi-y|$ close to γ_0.

To get good estimates and use Schur's lemma, we have to integrate by parts in y, in the direction parallel to the level sets of the function Φ, which is

$$V_2 := \nabla_y^{\perp}\Phi(x,y).$$

The conditions $|\Phi(x,y)| + |\Phi(\xi,y)| \lesssim 2^p$ show that $x-\xi$ is basically in the direction of $\nabla_x^{\perp}\Phi(x,y)$. We need that

$$V_2[y \to (\Phi(x,y) - \Phi(\xi,y))] \approx \nabla^2\Phi_{x,y}[(x-\xi), \nabla_y^{\perp}\Phi(x,y)]$$

is large, which leads to the nondeneracy condition

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The quasilinear I-method can be used to control the growth of the high order energy weighted norms

$$\|u\|_{H^{N_0}} \quad \text{and} \quad \|u\|_{H^{N_1}_{\Omega}} := \sup_{b\in[0,N_1]} \|\Omega^b u\|_{L^2}$$

where $\Omega = x_1\partial_2 - x_2\partial_1$ is the rotation vector-field.

To prove dispersion we use the following bootstrap proposition:

Proposition: Assume that U is a solution on some time interval $[0, T]$, with initial data U_0. Define, as before, $V(t) = e^{it\Lambda}U(t)$. Assume that

$$\|U_0\|_{H^{N_0}\cap H^{N_1}_\Omega} + \|V_0\|_Z \leq \varepsilon_0 \ll 1$$

and

$$(1+t)^{-p_0}\|U(t)\|_{H^{N_0}\cap H^{N_1}_\Omega} + \|V(t)\|_Z \leq \varepsilon_1 \ll 1$$

for all $t \in [0, T]$. Then, for any $t \in [0, T]$

$$\|V(t)\|_Z \lesssim \varepsilon_0 + \varepsilon_1^2.$$

The choice of the Z-norm is crucial. Examples of Z-norms used in semilinear analysis: the Strichartz norms, the $X^{s,b}$ spaces, Tataru's spaces.

In our (weighted) case we use Duhamel formula and the concept of space-time resonances (Germain–Masmoudi–Shatah):

$$(\partial_t + i\Lambda)U = \sum_{\pm} \mathcal{N}(U_\pm, U_\pm),$$

where the nonlinearities are defined by

$$(\mathcal{F}\mathcal{N}(f,g))(\xi) = \int_{\mathbb{R}^2} \mathfrak{m}(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta.$$

With $V(t) = e^{it\Lambda}U(t)$, the Duhamel formula is

$$\widehat{V}(\xi,t) = \widehat{V}(\xi,0) + \sum_{\pm}\int_0^t e^{is\Phi(\xi,\eta)}\mathfrak{m}(\xi,\eta)\widehat{V_\pm}(\xi-\eta,s)\widehat{V_\pm}(\eta,s)\,d\eta ds.$$

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Critical points (spacetime resonances): with

$$\Phi(\xi, \eta) = \Lambda(\xi) \pm \Lambda(\eta) \pm \Lambda(\xi - \eta)$$

the set of space-time resonances is

$$\{(\xi, \eta) : \Phi(\xi, \eta) = 0 \text{ and } \nabla_\eta \Phi(\xi, \eta) = 0\}.$$

In our case

$$(\xi, \eta) = (\gamma_1 \omega, \gamma_1 \omega / 2),$$

where $\omega \in \mathbb{S}^1$ and $\gamma_1 = \sqrt{2}$.

We define

$$Q_{jk}f := \varphi_j(x)P_k f(x).$$

We define

$$\|f\|_Z := \sup_{(k,j)\in\mathcal{J}} \sup_{|\alpha|\leq 50,\, m\leq N_1/2} \|D^\alpha \Omega^m Q_{jk} f\|_{B_j^\sigma},$$

where

$$\|g\|_{B_j^\sigma} := 2^{(1-50\delta)j} 2^{-(1/2-49\delta)n} \|A_n g\|_{L^2}.$$

The operators A_n are projection operators relative to the location of the spheres of space-time resonances, $||\xi| - \gamma_1| \approx 2^{-n}$.

Our Z norm depends in a significant way on both the linear part of the operator and the quadratic part of the equation. Norms of this type were introduced in work on the Euler–Maxwell equations in 3d.

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Resonances for bilinear interactions:

- time resonance

$$\Phi(\xi, \eta) = 0;$$

- space-time resonance (dispersive analysis)

$$\Phi(\xi, \eta) = 0 \quad \text{and} \quad \nabla_\eta \Phi(\xi, \eta) = 0 \quad \text{and} \quad \nabla_\xi \Phi(\xi, \eta) \neq 0;$$

- nondegenerate space-time resonance (dispersive analysis)

$$\nabla_\eta^2 \Phi(\xi, \eta) \quad \text{non-singular at space-time resonances.}$$

- restricted nondegenerate time resonance (energy method)

$$\Phi(\xi, \eta) = 0, \qquad \Upsilon(\xi, \eta) := \nabla^2_{\xi,\eta} \Phi(\xi, \eta) \left[\nabla_\xi^\perp \Phi(\xi, \eta), \nabla_\eta^\perp \Phi(\xi, \eta) \right] \neq 0.$$

Some open problems

1. Construction of long-term solutions with dynamically nontrivial vorticity:

$$T_{existence} \approx \Big(\frac{1}{|vorticity|}\Big)^p$$

A theorem of this type for the Euler–Maxwell one-fluid plasma model was proved by I.–Lie, for $p = 1$. Work in progress with O. Pocovnicu.

2. Dynamical formation of singularities in the two-fluid interface model. The "splash" singularity of Castro–Córdoba–Fefferman–Gancedo–Gómez-Serrano cannot form in the case of two-fluid interfaces. A possible scenario is self-intersection of the interface and loss of regularity at the same point.

3. Construction of global solutions in the case of two-fluid interfaces with suitable parameters $\rho_1, \rho_2, \sigma, g > 0$.

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$$T_{existence} \approx \Big(\frac{1}{|vorticity|}\Big)^{p}$$

A theorem of this type for the Euler–Maxwell one-fluid plasma model was proved by I.–Lie, for $p = 1$. Work in progress with O. Pocovnicu.

2. Dynamical formation of singularities in the two-fluid interface model. The "splash" singularity of Castro–Córdoba–Fefferman–Gancedo–Gómez-Serrano cannot form in the case of two-fluid interfaces. A possible scenario is self-intersection of the interface and loss of regularity at the same point.

3. Construction of global solutions in the case of two-fluid interfaces with suitable parameters $\rho_1, \rho_2, \sigma, g > 0$.

Work of Xuecheng Wang

1. Construction of global solutions of the gravity water wave model in 2d (1d interface), with no momentum condition on the velocity field, i.e. infinite energy. This improves on earlier work of Wu (almost global solutions), I.–Pusateri, Alazard–Delort, and Ifrim–Tataru. All earlier theorems required

$$|\nabla|^{-1/2} v(h(x,t),t) \in L^2(\mathbb{R}).$$

Wang removes this condition, which requires infinite energy solutions.

The construction of Wu of an energy that has a quartic "bulk" appears to fail. Wang uses the full quasilinear I-method to prove an energy inequality of the form

$$|\mathcal{E}_N(t) - \mathcal{E}_N(0)| \lesssim \text{Cubic term with special structure} \\ + \text{Quartic term.}$$

2. Construction of global solutions of the gravity water wave model in 3d (2d interface), with flat finite bottom. The corresponding theorem with infinite depth was proved by Germain–Masmoudi–Shatah and Wu. The linear dispersion in the finite bottom case is

$$\Lambda(\xi) = \sqrt{|\xi| \tanh(|\xi|)}.$$

The energy part of the proof is similar to the infinite depth problem (the solution still has $1/t$ decay). To prove dispersion, particularly for frequencies close to 0, he uses the Z-norm method with a well constructed norm.