Universality in Compressed Sensing

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Outline

1. Compressed Sensing
2. The universality problem
3. Proof outline
4. An interesting phenomenon
5. Conclusion

Compressed Sensing

Compressed Sensing

$$y = A\,x_0 + w$$

Estimate $x_0 \in \mathbb{R}^N$ given $y \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times N}$.

Compressed Sensing

$$\boldsymbol{y} = \boldsymbol{A}\,\boldsymbol{x}_0 + \boldsymbol{w}$$

Estimate $\boldsymbol{x}_0 \in \mathbb{R}^N$ given $\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{n\times N}$.

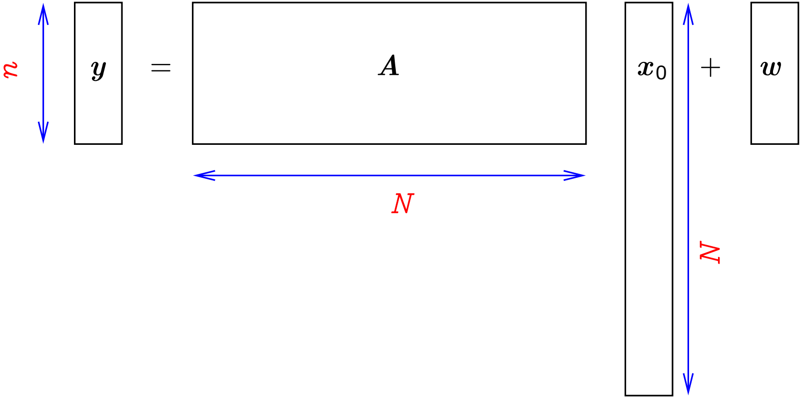
Classical case: $n > N$

$$\boldsymbol{y} = \boldsymbol{A}\,\boldsymbol{x}_0 + \boldsymbol{w}$$

$$\widehat{\boldsymbol{x}}(\boldsymbol{y}, \boldsymbol{A}) = \arg\min_{\boldsymbol{x} \in \mathbb{R}^N} \ \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|_2^2$$

[Legendre, 1805; Gauss 1809]

Compressed Sensing: $n < N$



Candes, Donoho, Tao, Tropp, Indyk, Gilbert, . . . , 2006-. . . 10^4 papers

$n < N$?

Key assumption: x_0 is sparse!

Example:

$$\|x_0\|_0 = |\mathrm{supp}(x_0)| = k \ll N$$

$n < N$?

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Example:

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Method of choice

LASSO/Basis pursuit denoising:

$$\widehat{x}(y, A) = \arg\min_{x \in \mathbb{R}^N} \left\{ \frac{1}{2} \|y - Ax\|_2^2 + \lambda \|x\|_1 \right\}$$

[Tibshirani 1996; Chen, Donoho 1994]

Noiseless case: $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}_0$

Basis pursuit $(\lambda \to 0)$

$$\begin{aligned} &\text{minimize} && \|x\|_1\,, \\ &\text{subject to} && \boldsymbol{y} = \boldsymbol{A}x\,. \end{aligned}$$

Does this work?

Noiseless case: $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}_0$

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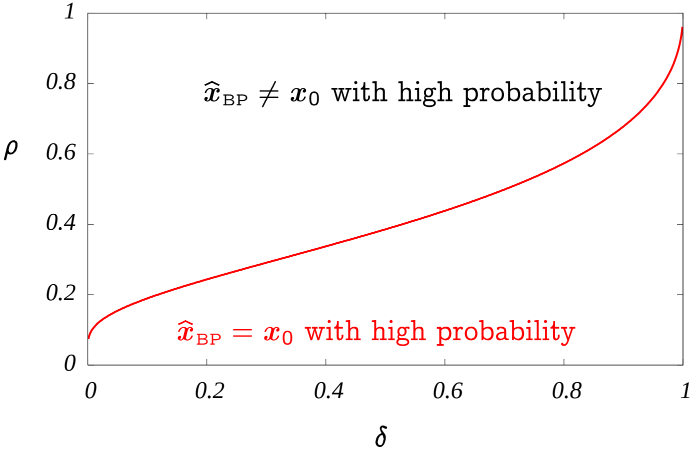
Does this work? Gaussian model

- $A_{ij} \sim_{i.i.d.} \mathrm{N}(0, 1/n)$
- $n/N \to \delta \in (0, 1)$
- $\|x_0\|_0/n \to \rho \in (0, 1)$

Phase diagram 1: 'ℓ_0–ℓ_1 equivalence' (noiseless)

$$N, n \to \infty, \qquad n/N = \delta, \qquad \|x_0\|_0/n = \rho$$



[Donoho 2006, Affentranger-Schneider 1992]

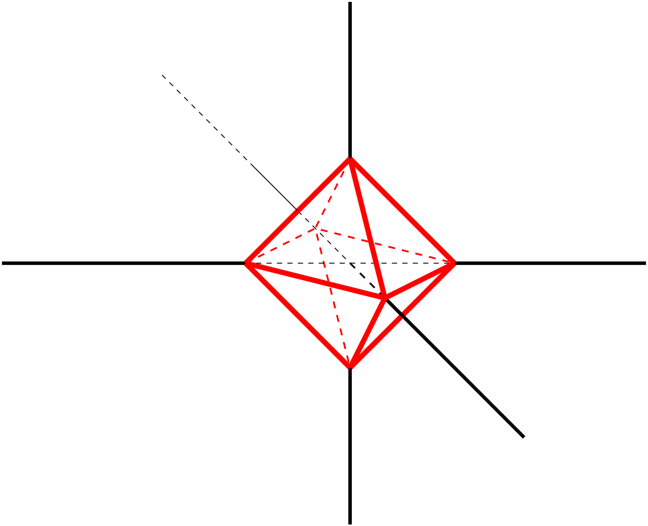
Phase transition curve

$$\delta = \frac{2\phi(\alpha)}{\alpha + 2(\phi(\alpha) - \alpha\Phi(-\alpha))}\,,$$
$$\rho = 1 - \frac{\alpha\Phi(-\alpha)}{\phi(\alpha)}\,.$$

$$\alpha \in [0,\infty), \qquad \phi(x) \equiv e^{-x^2/2}/\sqrt{2\pi}, \qquad \Phi(x) \equiv \int_{-\infty}^{x} \phi(z)\,\mathrm{d}z$$

Entr'acte: Geometric interpretation

ℓ_1 ball in $\mathbb{R}^N$



$\|x_0\|_0 = 1 \leftrightarrow$ vertices

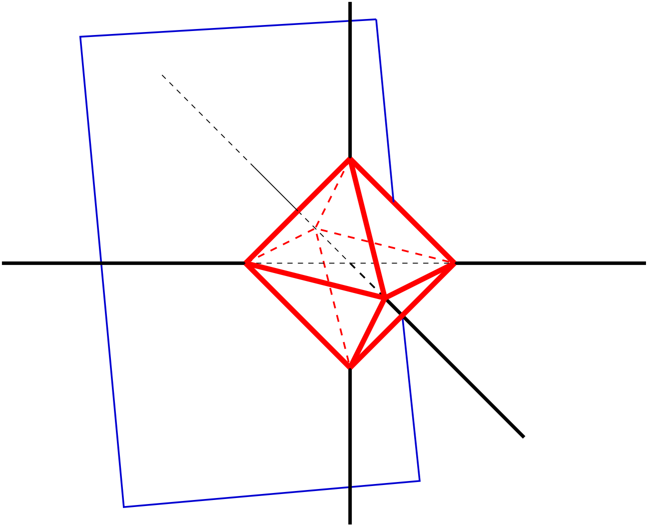
$\|x_0\|_0 = 2 \leftrightarrow$ edges

$\|x_0\|_0 = 3 \leftrightarrow$ 2-faces

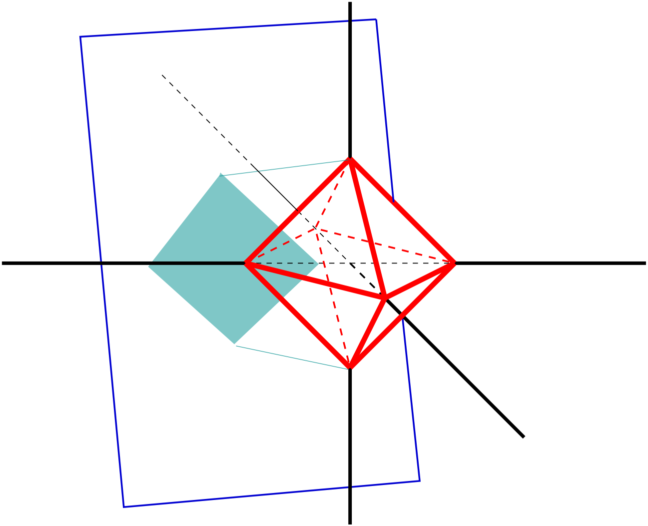
$\|x_0\|_0 = 4 \leftrightarrow$ 3-faces

...

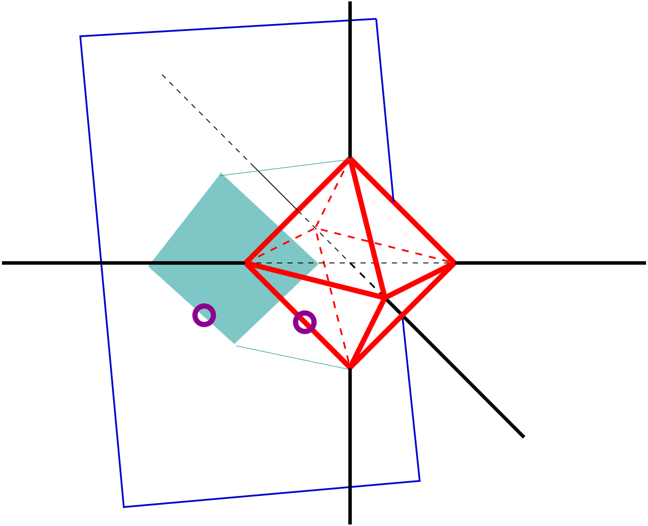
Random n-dimensional plane



Projection



Does it fall on the boundary?



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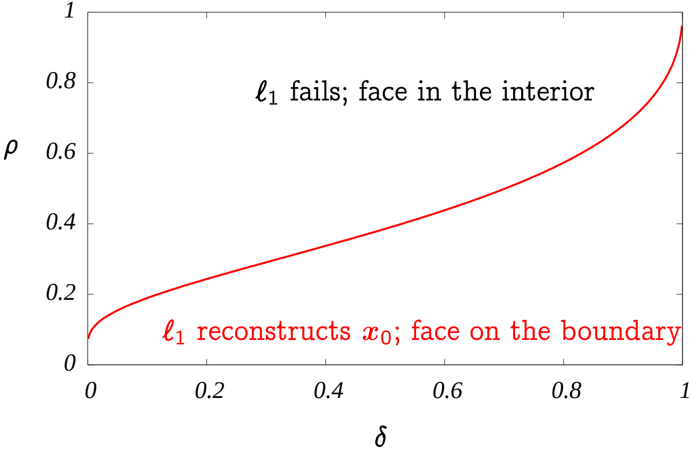


[Donoho 2006, Affentranger-Schneider 1992]

Geometric phase transition

- For $k \leq (\rho_c(\delta) - \varepsilon)\, n$, most k-faces fall on the boundary of the shadow.
- For $k \geq (\rho_c(\delta) + \varepsilon)\, n$, most k-faces fall on the boundary of the shadow.

The universality problem

Conjecture (Donoho, Tanner, 2009)

The above predictions are universal for a broad range of random matrices.

Donoho, Tanner, *Phil. Trans. R. Soc. A* (2009) **367**, 4273-4293

across a range of underlying matrix ensembles. We ran millions of linear programs using random matrices spanning several matrix ensembles and problem sizes; visually, the empirical phase transitions do not depend on the ensemble, and they agree extremely well with the asymptotic theory assuming Gaussianity. Careful statistical analysis reveals

Example: Random Fourier measurements

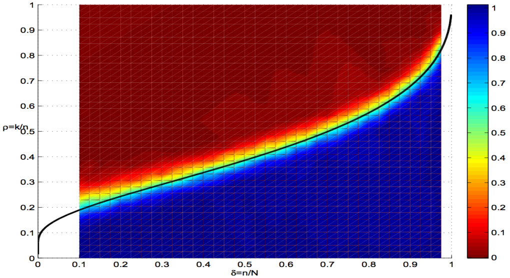
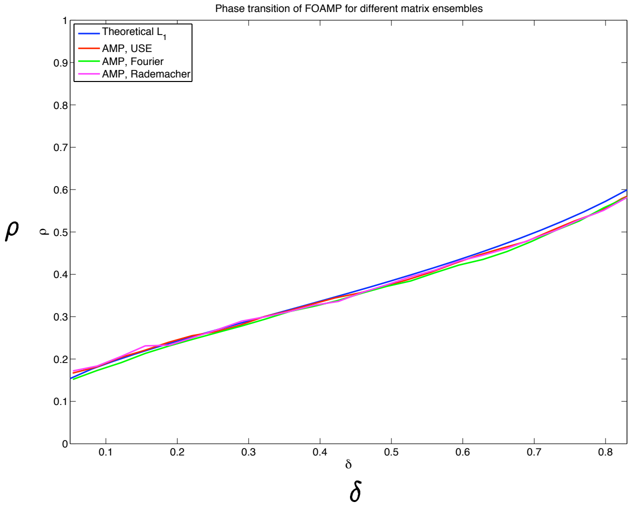


FIGURE 3. *Compressed Sensing from random Fourier measurements.* Shaded attribute: fraction of realizations in which $\ell_1$minimization (1.2) reconstructs an image accurate to within six digits. Horizontal axis: undersampling fraction $\delta = n/N$. Vertical axis: sparsity fraction $\rho = k/n$.

[Donoho, Tanner 2009]

Other examples



IID model

- A_{ij} i.i.d., $\mathbb{E}\{A_{ij}\} = 0$, $\mathbb{E}\{A_{ij}^2\} = 1/n$
- $n/N \to \delta \in (0, 1)$
- $\|x_0\|_0/n \to \rho \in (0, 1)$

Theorem (Bayati, Lelarge, Montanari, 2012)

Assume the IID model, with

- A_{ij} *subgaussian*
- $A_{ij} \stackrel{\rm d}{=} \widetilde{A}_{ij} + \varepsilon\, G_{ij}\,, \quad G_{ij} \sim {\rm N}(0,1)$

Then the $\ell_1 - -\ell_0$ phase transition is the same as for the Gaussian model. Namely

- *If $\rho < \rho_c(\delta)$, then $\mathbb{P}(\widehat{\boldsymbol{x}}(\boldsymbol{y}, \boldsymbol{A}) = \boldsymbol{x}_0) \to 1$,*
- *If $\rho > \rho_c(\delta)$, then $\mathbb{P}(\widehat{\boldsymbol{x}}(\boldsymbol{y}, \boldsymbol{A}) = \boldsymbol{x}_0) \to 0$.*

Proof outline

Two possible strategies

- **Lindeberg method:** Replace the A_{ij}'s one-by-one.

  [Korada, Montanari 2011]

  [weaker result]

- **Polynomial approximation**: Approximate function by polynomial

  [Bayati, Lelarge, Montanari 2012]

Two possible strategies

- **Lindeberg method:** Replace the A_{ij}'s one-by-one.

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- **Polynomial approximation**: Approximate function by polynomial

  [Bayati, Lelarge, Montanari 2012]

Polynomial approximation

- $\boldsymbol{X} = (X_1, \dots, X_m) \sim_{i.i.d.} P_X$
- $\boldsymbol{Y} = (Y_1, \dots, Y_m) \sim_{i.i.d.} P_Y$
- $\mathbb{E}\{X_i\} = \mathbb{E}\{Y_i\}$, $\mathbb{E}\{X_i^2\} = \mathbb{E}\{Y_i^2\}$
- $F : \mathbb{R}^m \to \mathbb{R}$

Want to prove

$$\mathbb{E}\{F(\boldsymbol{X})\} \approx \mathbb{E}\{F(\boldsymbol{Y})\}$$

Prove

$$F(x) \approx \sum_{\boldsymbol{k} \in \{0,1,2\}^m} F_{\boldsymbol{k}} x^{\boldsymbol{k}} \, ,$$

$$x^{\boldsymbol{k}} \equiv \prod_{i=1}^{m} x_i^{k_i} \, .$$

Polynomial approximation

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In our case

- $\boldsymbol{A} = (A_{ij})_{i \le n, j \le N}$

$$F(\boldsymbol{A}) = \mathbb{I}\Big(\boldsymbol{x}_0 = \arg\min\big\{\|\boldsymbol{x}\|_1 \ : \ \boldsymbol{A}(\boldsymbol{x} - \boldsymbol{x}_0) = 0\big\}\Big)$$

First simplification: Can fix

$$x_0 = (\underbrace{1, 1, \dots, 1}_{k \text{ non-zeros}}, 0, 0, \dots, 0)$$

How to approximate it?

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How to approximate it?

Will focus on

- If $\rho < \rho_c(\delta)$, then $\mathbb{P}(\widehat{\boldsymbol{x}}(\boldsymbol{y}, \boldsymbol{A}) = \boldsymbol{x}_0) \to 1$

Optimality condition

$$x_0 \in \arg\min \{ \|x\|_1 \quad : \quad A(x - x_0) = 0 \}$$

...if and only if there exists

$$v \in \partial \|x_0\|_1 \; : \qquad v \in \mathrm{Image}(A^{\mathsf{T}})$$

Subdifferential

$$\partial \|x_0\|_1 \equiv \Big\{ v \in \mathbb{R}^N : v_i = \mathrm{sign}(x_{0,i}),\, i \in \mathrm{supp}(x_0), \\ |v_i| \le 1,\, i \notin \mathrm{supp}(x_0) \Big\}$$

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Quantitative version

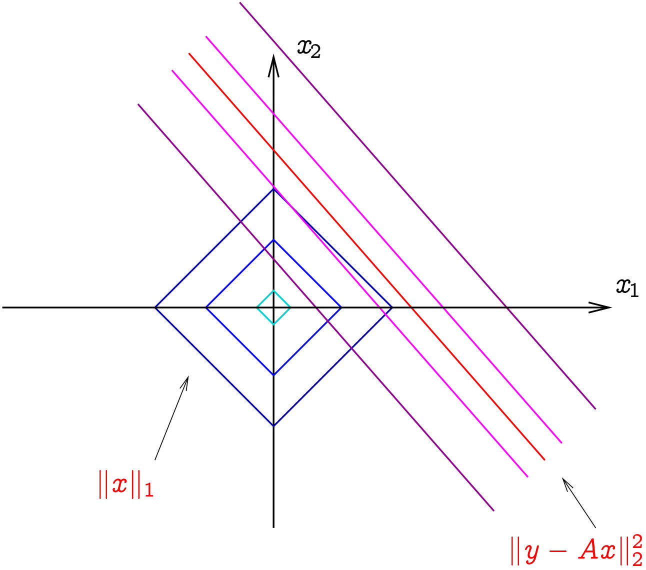
Lemma

Assume

1. *There exists* $v \in \partial\|x_0\|_1$ *and* $z \in \mathbb{R}^n$ *with* $v = A^{\mathsf{T}} z + w$ *and* $\|w\|_2 \le \sqrt{n}\,\varepsilon$, *with* $\varepsilon \le \varepsilon_0$.
2. *For* $c \in (0,1)$, *let* $S(c) \equiv \{i \in [n] : |v_i| \ge 1 - c\}$. *Then, for any* $S' \subseteq [n]$, $|S'| \le c_1 n$, $\sigma_{\min}(A_{S(c_1)\cup S'}) \ge c_2$.
3. $c_3^{-1} \le \sigma_{\max}(A)^2 \le c_3$.

Then $x_0 = \arg\min\{\|x\|_1 \;\; : \;\; A(x - x_0) = 0\}$

Where is the problem



Plan

- Construct an approximate subgradient $\boldsymbol{v}$
- Check properties $1, 2, 3$

Universality

$$\boldsymbol{v} \approx \sum_{\boldsymbol{k} \in \{0,1,2\}^{n \times N}} \boldsymbol{v}_{\boldsymbol{k}} \, \boldsymbol{A}^{\boldsymbol{k}}$$

Idea: Use an iterative algorithm to construct $\boldsymbol{v}$

‘Approximate Message Passing’ (AMP)

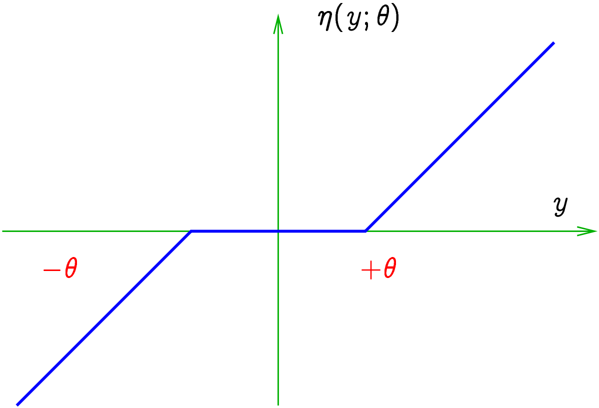
Time index $t \in \{0, 1, 2, \dots, T\}$

$$\begin{aligned} \boldsymbol{x}^{t+1} &= \eta(\boldsymbol{x}^t + \boldsymbol{A}^{\mathsf{T}} \boldsymbol{z}^t; \theta_t) \\ \boldsymbol{z}^t &= \boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^t + \mathsf{b}_t \boldsymbol{z}^{t-1} \end{aligned}$$

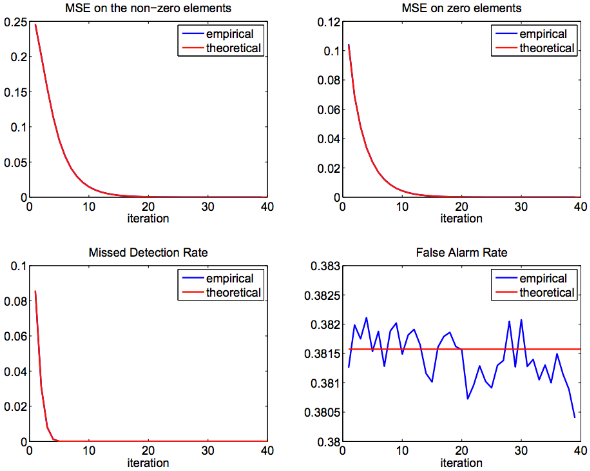
- Algorithm parameters: $\theta_t, \mathsf{b}_t \in \mathbb{R}$
- Soft thresholding function: $\eta(\,\cdot\,; \theta) : \mathbb{R} \to \mathbb{R}$

[Donoho, Maleki, Montanari, 2009]

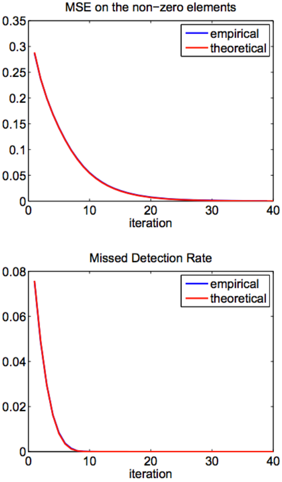
η



Algorithm evolution: $\delta = 0.2$, $\rho = 0.3$



Algorithm evolution: $\delta = 0.15$, $\rho = 0.3$



Asymptotic analysis

T fixed, $N, n \to \infty$

Dynamics + Randomness + High dimension

'State evolution': T fixed, $N, n \to \infty$

Time index $t \in \{0, 1, 2, \dots, T\}$

$$x^{t+1} = \eta(y^t; \theta_t)$$
$$y^t = x^t + A^T z^t$$
$$z^t = y - Ax^t + \mathsf{b}_t z^{t-1}$$

Theorem (Bayati, Montanari, 2010)

For A Gaussian (letting $X_0 \sim p_{X_0}$ independent of $Z \sim \mathrm{N}(0,1)$)

$$y^t \approx \mathrm{N}(x_0, \sigma_t^2 \mathrm{I}_{N\times N}),$$
$$\sigma_{t+1}^2 = \mathbb{E}\Big\{\big[\eta(X_0 + \sigma_t Z; \theta_t) - X_0\big]^2\Big\}$$

Construction of the subgradient

$$\boldsymbol{x}^{t+1} = \eta(\boldsymbol{x}^t + \boldsymbol{A}^{\mathsf{T}}\boldsymbol{z}^t; \theta_t)$$
$$\boldsymbol{z}^t = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^t + \mathsf{b}_t \boldsymbol{z}^{t-1}$$

Note: $|\eta(x;\theta) - x| \le \theta$

$$v_i^t = \begin{cases} \text{sign}(x_{0,i}) & \text{if } i \in S\,, \\ \frac{1}{\theta_{t-1}}\big(x^{t-1} + A^{\mathsf{T}}z^{t-1} - \eta(x^{t-1} + A^{\mathsf{T}}z^{t-1}; \theta_{t-1})\big)_i & \text{otherwise,} \end{cases}$$

Construction of the subgradient

$$x^{t+1} = \eta(x^t + A^\top z^t; \theta_t)$$
$$z^t = y - Ax^t + \mathsf{b}_t z^{t-1}$$

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Proof strategy

- Analyze AMP for $\boldsymbol{A}$ non-gaussian
  - Approximate η by a polynomial
  - $\Rightarrow \boldsymbol{x}^t$, $\boldsymbol{z}^t$, $\boldsymbol{v}^t$ =Polynomials $(\{A_{ij}\})$
  - Use moment method
  - $\Rightarrow$ state evolution

- Check condition for optimality.

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An interesting phenomenon

Let us compare

$$\boldsymbol{x}^{t+1} = \eta(\boldsymbol{x}^t + \boldsymbol{A}^{\mathsf{T}}\boldsymbol{z}^t;\theta_t)$$
$$\boldsymbol{z}^t = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^t + \mathsf{b}_t\boldsymbol{z}^{t-1}$$

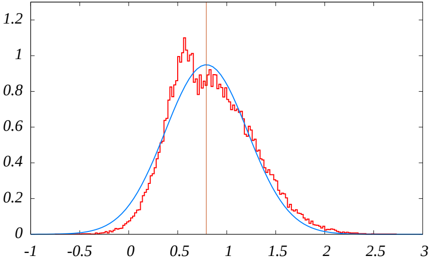
and

$$\boldsymbol{x}^{t+1} = \eta(\boldsymbol{x}^t + \boldsymbol{A}^{\mathsf{T}}\boldsymbol{z}^t;\theta_t)$$
$$\boldsymbol{z}^t = \boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}^t$$

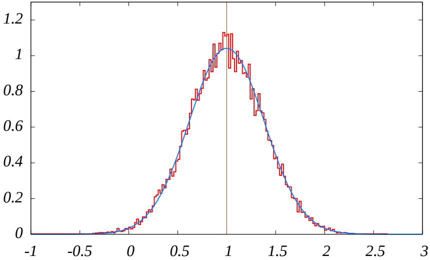
'Onsager reaction term' (spin glass theory)

What's the big deal with $-\mathrm{b}_t z^{t-1}$?

Distribution of $(x^t + A^T z^t)_i$ conditional on $x_{0,i} = 1$



without



with

Let us look at a simpler example (symmetric)

- $\boldsymbol{A} \in \mathbb{R}^{n \times n}$
- $A_{ii} = 0$, $\{A_{ij}\}_{i<j}$ iid, $\mathbb{E}\{A_{ij}\} = 0$, $\mathbb{E}\{A_{ij}^2\} = 1/n$
- $f : \mathbb{R} \to \mathbb{R}$, $f(\boldsymbol{x}) = (f(x_1), f(x_2), \dots, f(x_n))$

$$\boldsymbol{x}^{t+1} = \boldsymbol{A} f(\boldsymbol{x}^t) - \mathsf{b}_t f(\boldsymbol{x}^{t-1})$$

$$\mathsf{b}_t = \frac{1}{n} \sum_{i=1}^{n} f'(x_i^t)$$

Special case: TAP equatios for mean field spin glasses

[Bolthausen 2011]

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Special case: TAP equatios for mean field spin glasses

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Symmetric case

$$x^{t+1} = Af(x^t) - \mathsf{b}_t f(x^{t-1})$$

$$\mathsf{b}_t = \frac{1}{n} \sum_{i=1}^{n} f'(x_i^t)$$

Theorem (Bayati, Montanari, 2010)

If $A \sim \text{GOE}(n)$, $f \in \text{Lip}(\mathbb{R})$, *then (here* $Z \sim \text{N}(0,1)$*)*

$$x^t \approx \text{N}(0, \sigma_t^2 \text{I}_{n\times n}), \qquad \sigma_{t+1}^2 = \mathbb{E}\{f(\sigma_t Z)^2\}.$$

Theorem (Bayati, Lelarge, Montanari, 2012)

If $A \sim \text{Wigner}(n)$, $f \in \text{Poly}(\mathbb{R})$, *then (here* $Z \sim \text{N}(0,1)$*)*

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$$x^t \approx \mathrm{N}(0, \sigma_t^2 \mathrm{I}_{n\times n}), \qquad \sigma_{t+1}^2 = \mathbb{E}\{f(\sigma_t Z)^2\}.$$

Symmetric case

$$x^{t+1} = Af(x^t) - \mathsf{b}_t f(x^{t-1})$$
$$\mathsf{b}_t = \frac{1}{n}\sum_{i=1}^{n} f'(x_i^t)$$

Theorem (Bayati, Montanari, 2010)

If $A \sim \mathsf{GOE}(n)$, $f \in \mathsf{Lip}(\mathbb{R})$, *then (here* $Z \sim \mathrm{N}(0,1)$*)*

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Even simpler: $f(x) = x$

Say $\boldsymbol{x}^0 = \mathbf{1}$, $\boldsymbol{x}^{-1} = 0$

$$\boldsymbol{x}^{t+1} = \boldsymbol{A}\boldsymbol{x}^t - \boldsymbol{x}^{t-1}$$

Corollary

For al $t \geq 1$ fixed, as $n \to \infty$,

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$$x^2 = \sum_{j_1 \in [n] \setminus i, j_2 \in [n] \setminus j_1} A_{ij_1} A_{j_1,j_2} - 1$$

$$= \sum_{j_1 \in [n] \setminus i, j_2 \in [n] \setminus \{j_1, i\}} A_{ij_1} A_{j_1,j_2} + \sum_{j_1 \in [n] \setminus i} A_{ij_1} A_{j_1 i} - 1$$

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&\approx \sum_{j_1 \in [n] \setminus i, j_2 \in [n] \setminus \{j_1, i\}} A_{ij_1} A_{j_1, j_2} && \Rightarrow \mathrm{N}(0,1)
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$$\boldsymbol{x}^{t+1} = \boldsymbol{A}\boldsymbol{x}^t - \boldsymbol{x}^{t-1}$$

$$x_i^t \approx \sum_{\gamma \in \mathsf{NRW}(i,t)} \prod_{k=1}^{t-1} A_{\gamma(k),\gamma(k+1)} \quad \Rightarrow \mathrm{N}(0,1)$$

$$\mathsf{NRW}(i;t) \equiv \Big\{\gamma : \{0,\dots,t\} \to [n] : \quad \gamma(0) = i,\ \ \gamma(k-1) \neq \gamma(k+1)\ \forall k\Big\}$$

[$\Rightarrow$ by moment method]

General $f(x) =$poly(x)

Say $\boldsymbol{x}^0 = \mathbf{1}$, $\boldsymbol{x}^{-1} = 0$

$$\boldsymbol{x}^{t+1} = \boldsymbol{A} f(\boldsymbol{x}^t) - \mathsf{b}_t f(\boldsymbol{x}^{t-1})$$

$$x_i^t \approx \sum_{T \in \text{Trees}} c_f(T;t) \sum_{\gamma \in \mathsf{NRT}(i;t)} \prod_{e \in E(T)} A_{\gamma(e)} \quad \Rightarrow \mathrm{N}(0, \sigma_t^2)$$

$$\mathsf{NRT}(i;t) \equiv \Big\{ \gamma : V(T) \to [n] : \quad \gamma(\text{root}) = i, \\ \gamma(\text{desc}(v)) \neq \gamma(\text{parent}(v)) \; \forall v \Big\}$$

Conclusion

Open problems

- Noisy compressed sensing
- Other matrix ensembles
  (i.i.d. entries, i.i.d. rows, Fourier sections,...)
- Other convex optimization problems
- Universality of AMP (non-polynomial functions)

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