The stochastic Landau-Lifshitz equation

A. de Bouard

CMAP, Ecole Polytechnique, France
joint work with F. Alouges and A. Hocquet (CMAP)

New challenges in PDE: Deterministic dynamics and randomness in high and infinite dimensional systems
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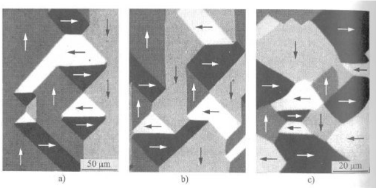
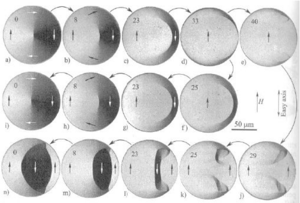
Modeling of micromagnetics

- Pierre Weiss, 1907 : domain theory

  Magnetic materials = collection of uniform magnets minimizing magnetostatic + jump energy





From A. Hubert and R. Schäfer, Magnetic domains

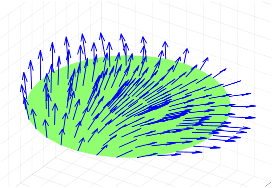
- W.F. Brown, 1940-60 : continuous model

Magnetization (magnet distribution)

$M : \mathcal{O} \to \mathbf{R}^3$

with

$|M(x)| = M_s(T)$ a.e. $x \in \mathcal{O}$



here T = temperature

$m = M/M_s(T)$ minimizes the Brown energy (non dimensional form) :

$$\mathcal{E}(m) = \frac{1}{2}\int_{\mathcal{O}} |\nabla m|^2\, dx - \int_{\mathcal{O}} H_{ext} \cdot m\, dx - \frac{1}{2}\int_{\mathcal{O}} H_d(m) \cdot m\, dx + K \int_{\mathcal{O}} G(m)\, dx$$

with $m(x) \in S^2$, p.p.

- $\frac{1}{2}\int_{\mathcal{O}} |\nabla m|^2 \, dx$: exchange energy
- $-\int_{\mathcal{O}} H_{ext} \cdot m \, dx$: external energy (due to external field H_{ext})
- $-H_d(m)$: stray field (magnetic field induced by the particle itself) ;

$$H_d(m) = -\nabla\Delta^{-1}\text{div } \bar{m}$$

- $K\int_{\mathcal{O}} G(m) \, dx$: anisotropic energy

$\rightsquigarrow$ minimization of $m(x)$ subject to constraint $|m(x)| = 1$, a.e. leads to a frustrated system

- Lots of theoretical/asymptotic studies (depending on size/material)
- Strong link with harmonic maps into the sphere

Dynamical equation

Landau-Lifshitz $\sim$ 1935 :

$$H_{eff}(m) = -D_m\mathcal{E}(m) = \Delta m + H_{ext} + H_d(m) - KD_mG(m)$$

then the LL equation is given by

$$\begin{cases} \dfrac{\partial m}{\partial t} = m \times H_{eff}(m) - \alpha m \times (m \times H_{eff}(m)) \text{ in} \mathcal{O} \\ m(0,x) = m_0(x) \text{ in } \mathcal{O} \\ \dfrac{\partial m}{\partial n} = 0 \text{ on } \partial\mathcal{O} \end{cases}$$

$\alpha > 0$, damping :

$$\frac{d\mathcal{E}(m)}{dt} = -H_{eff}(m) \cdot \frac{\partial m}{\partial t} = -\alpha |m \times H_{eff}(m)|^2$$

Nonzero temperature : thermal activation

$\rightsquigarrow$ add to H_{eff} a random field H_{therm} (Brown, 1963)

Usually for physicists :

- H_{therm} gaussian
- $\langle H_{therm}(x,t)\rangle = 0$
- $\langle H_{therm}(x,t)\cdot H_{therm}(y,s)\rangle = \mu\delta_{x-y}\delta_{t-s}$

$\rightsquigarrow$ space-time white noise

Actually : ideal case

Time correlation length $\sim 10^{-13}s$. (room temperature)

Response time of the medium $\sim 10^{-10}s$.

and independence of the single domain particles

Finite dimensional case

Single particle domain :

Very small objects ($\sim 10nm$) $\rightsquigarrow$ neglect exchange energy :

$$m(t,x) = m(t)\mathbb{1}_{\mathcal{O}}(x)$$

$\rightsquigarrow$ Stoner Wohlfarth model :

$$\frac{dm}{dt} = m \times \tilde{H}_{eff} - \alpha m \times (m \times \tilde{H}_{eff})$$

with $m(t) \in S^2 \subset \mathbf{R}^3$,

$$\tilde{H}_{eff} = H_{ext} - Dm - K\nabla_m G(m) + H_{therm},$$

$D = (D_{ij})_{1\leq i,j\leq 3}$ positive definite matrix, $H_{therm} = \sqrt{\frac{2\varepsilon\alpha}{1+\alpha^2}}\dot{W}(t)$, $W(t) = (W_1(t), W_2(t), W_3(t))$, three dimensional Brownian motion

Remarks :

- Equivalent equation (same law) :

$$\frac{dm}{dt} = m \times (H_{eff} + \sqrt{2\varepsilon\alpha}\dot{W}(t)) - \alpha m \times (m \times H_{eff})$$

- Stratonovich product : $|m(t)| = 1$, a.e.
- Existence and uniqueness of (global in time) strong solutions is clear (compact state space)
- $m \times dW = dB$ additive Brownian motion with values on the sphere

Well studied : Brown (1962) ; Garcia-Palacios-Lazaro (1998) ; Kohn-Reznikov-Vanden-Eijnden (2005) ; Bañas-Neklyudov-Prohl (2013), $\cdots$

- Existence and uniqueness of invariant measure (for usual anisotropic energies) : Gibbs measure

$$\mu(dm) = Ze^{-\mathcal{E}(m)/\varepsilon}dm, \quad Z = \int_{S^2} e^{-\mathcal{E}(m)/\varepsilon}dm$$

- Use of Large Deviation Principle to explain influence of T on Stoner-Wohlfarth astroïds, hysteresis loops, ...
- Case of N single particles, coupled thanks to exchange energy :

$$M = (m_1, \cdots, m_N),\ m_j \in S^2 \in \mathbf{R}^3,\ J = (J_{kl})_{1\leq k,l\leq N},$$

where J is sym. def. ≥ 0 ; then exchange energy given by

$$\mathcal{E}_{exch}(M) = \frac{1}{2}\sum_{k;l=1}^{N} J_{kl}\langle m_k, m_l\rangle.$$

Infinite dimensional case

- Three dimensional situation : no hope for space-time white noise
- Deterministic case ($H_{therm} = 0$) :
  - Global existence of weak solutions (no uniqueness)
  - Local existence of strong solutions (blow up ?)

Formally : $\varphi : \mathcal{O} \to \mathbf{R}^3$ regular

$$\langle m \times \Delta m, \varphi \rangle = \langle \nabla (m \times \nabla m), \varphi \rangle = \langle \nabla m, m \times \nabla \varphi \rangle$$

$\rightsquigarrow$ natural space for weak solutions : $\mathbf{H}^1 = (H^1(\mathcal{O}))^3$

In what follows :

$$H_{eff}(m) = \Delta m; \quad H_{therm} = h(x)\dot{W}(t), \quad h : \mathcal{O} \to \mathbf{R}^3$$

and $W(t)$ is a one dimensional Brownian motion ; can actually treat more general situations

$$(SLL) \quad \partial_t m = m \times (\Delta m + h\dot{W}) - \alpha m \times (m \times (\Delta m + h\dot{W}))$$

Definition : Weak solutions (Brzezniak-Goldys-Jegaraj)

$(\Omega, \mathcal{F}, \mathbf{P}, (\mathcal{F}_t)_{t\geq 0}, W, m)$ such that for all $T > 0$,

- $m \in C([0,T]; H^{-1}(\mathcal{O}))$, $\mathbf{P}$.a.s.
- $\mathbf{E}\left(\sup_{t\leq T} |\nabla m(t)|^2_{L^2(\mathcal{O})}\right)$ is finite
- $|m(t,x)| = 1$, $dx \otimes \mathbf{P}$ a.e.
- m satisfies for all $\varphi \in C^\infty(\bar{\mathcal{O}}, \mathbf{R}^3)$,

$$\langle m(t), \varphi\rangle - \langle m_0, \varphi\rangle = -\int_0^t \langle \nabla m, (\nabla\varphi) \times m\rangle \, ds$$
$$-\alpha \int_0^t \langle \nabla m, \nabla(m \times \varphi) \times m\rangle + \int_0^t \langle G(m)h, \varphi\rangle \circ dW(s)$$

with $G(m)h = m \times h - \alpha m \times (m \times h)$.

Theorem (Brzezniak-Goldys-Jegaraj, 2013) :

Let $m_0 \in \mathbf{H}^1$ with $|m_0(x)| = 1$, a.e. and $h \in (L^\infty \cap W^{1,3}(\mathcal{O}))^3$;
then there is a weak solution $(\Omega, \mathcal{F}, \mathbf{P}, (\mathcal{F}_t)_{t\geq 0}, W, m)$ of (SLL) ;

Moreover, for all $T > 0$,

$$\mathbf{E} \int_0^T |m \times \Delta m|_{L^2}^2 \, dt < +\infty,$$

and $m \in C^\alpha([0, T]; \mathbf{L}^2(\mathcal{O}))$ a.s. for any $\alpha < 1/2$.

Then m also satisfies

$$m(t) = m_0 + \int_0^t m \times \Delta m \, ds \\ -\alpha \int_0^t m \times (m \times \Delta m) \, ds + \int_0^t G(m) h \circ dW(s)$$

- Strategy of proof uses Galerkin approximation, tightness of the sequence and convergence of the associated martingale problem, together with representation theorems
- Nothing is known about the long time behaviour (invariant measures) : if

$$\mathcal{E}(m) = \frac{1}{2}\int_{\mathcal{O}} |\nabla m|^2 \, dx$$

then

$$\mathbf{E}(\mathcal{E}(m(t))) + \mathbf{E}\int_0^T \int_{\mathcal{O}} |m \times \Delta m|^2 \, dxdt \leq CT;$$

however lack of uniform compactness in time

- More precise results in the 1-D case $\mathcal{O} = (0, 1)$
Goldys-Le Ngan-Tran : uniqueness of the $\mathbf{H}^1$ solution, application of the LDP to the study of possible switching
Bruned-Hairer-Zambotti : space-time white noise (regularity structures) in the over damped case

Numerical schemes

$$(SLL) \quad \begin{cases} dm = [m \times \Delta m - \alpha m \times (m \times \Delta m)]\, dt + m \times \circ dW \\ m(0) = m_0 \text{ in } \mathcal{O} \\ \partial_n m(t,x) = 0 \text{ on } \partial\mathcal{O} \end{cases}$$

Here W is a Q-Wiener process on $\mathbf{L}^2$, such that $W(t) \in \mathbf{H}^2$ (Q finite trace sym. positive op. with values in $\mathbf{H}^2$)

Finite elements approximation :

Let $(\mathbf{T}_h)_{h>0}$ a triangular mesh of $\mathcal{O}$; replace $\mathbf{H}^1(\mathcal{O})$ by $\mathbf{V}_h$, with

$$\mathbf{V}_h = \{\varphi \in \mathbf{H}^1,\ \varphi_{|_T} \in \mathbf{P}^1 \text{ for each } T \in \mathbf{T}_h\}$$

Then, replace the test function $\varphi \in \mathbf{H}^1$ by $\varphi \in \mathbf{V}_h$ in the definition of the weak solutions

Time discretization :

Midd-point discretization (Stratonovich product) Banas, Brzezniak, Neklyudov, Prohl :

Given $m^n \in \mathbf{V}_h$ (approximation of $m(n\delta t, \cdot)$), look for $m^{n+1} \in \mathbf{V}_h$ satisfying : $\forall \varphi \in \mathbf{V}_h$,

$$\langle m^{n+1} - m^n, \varphi \rangle_h + \alpha(\delta t) \langle m^{n+1/2} \times (m^{n+1/2} \times \Delta_h m^{n+1}), \varphi \rangle_h$$
$$-(\delta t)\langle m^{n+1/2} \times \Delta_h m^{n+1/2}, \varphi \rangle_h = \langle m^{n+1/2} \times \Delta_n W, \varphi \rangle_h$$

with

$$m^{n+1/2} = \frac{1}{2}(m^n + m^{n+1}) \text{ and } \Delta_n W = W((n+1)\delta t) - W(n\delta t).$$

Theorem (BBNP, 2014) : Let $T > 0$ fixed and $m_0 \in \mathbf{H}^1$; then for $\delta t = T/N$, the sequence $((m^n)_{0 \leq n \leq N})_{N \in \mathbf{N}}$ converges in law, up to a subsequence, to a weak solution of (SLL) as N goes to infinity (i.e. δt goes to zero).

Remarks :

- The constraint $|m^n(x)| = 1$ is satisfied a.e. (not the case if discretization of the Itô equ.)
- The scheme is nonlinearly implicit $\rightsquigarrow$ requires the resolution of a nonlinear pb at each time step (fixed point, or Newton)
- No uniqueness for m^{n+1} (random selection theorem)

Our aim :

Build a linearly implicit scheme that respects the constraint $|m^n(x)| = 1$, a.e. $\rightsquigarrow$ implies that $dm \perp m$ a.e. ; hence natural to search the increment in a space of functions $\perp m^n$

From now on take $\alpha = 1$ for simplicity

Ideas : (originates from deterministic scheme F. Alouges, 2008)

- Consider the nonlinear term coming from

$$m \times (m \times \Delta m) = -\Delta m - m|\nabla m|^2$$

then for any $\varphi \in C^\infty(\mathcal{O})$ with $\varphi(x) \perp m(t,x)$ a.s.,

$$\langle m \times (m \times \Delta m), \varphi\rangle = -\langle \Delta m, \varphi\rangle = \langle \nabla m, \nabla \varphi\rangle$$

$\rightsquigarrow$ term along $m(t,x)$ recovered as a Lagrange multiplier, thanks to the constraint

- For the nonlinear term $m \times \Delta m$: use of the Gilbert form of the LL equation :

$$(SLLG) \qquad \begin{aligned} & dm - m \times dm \\ & = 2(\Delta m + m|\nabla m|^2)dt + \tfrac{1}{2}(Id - m\times)(m \times \circ dW) \end{aligned}$$

Definition of the semi-discrete scheme :

Let $G = Q^{1/2}$ (assumed HS from $\mathbf{L}^2$ into $\mathbf{H}^2$) and $(e_k)_{k\in\mathbf{N}}$ c.o.s. of $\mathbf{L}^2$; define the (random) space :

$$\mathbf{W}_{\delta t,n} = \{\psi \in \mathbf{H}^1(\mathcal{O}),\ \psi(x) \perp m^n_{\delta t}(x)\}$$

where the r.v. $m^n_{\delta t}(x)$ is a approximation of $m(n\delta t, x)$. then

- Define $v^n = v^n_{\delta t} \in \mathbf{W}_{n,\delta t}$ as the solutions of the pb :
  $\forall \varphi \in \mathbf{W}_{n,\delta t}$,

$$\begin{aligned} & \langle v^n - m^n \times v^n, \varphi\rangle + 2(\delta t)\langle\nabla v^n, \nabla\varphi\rangle \\ (\mathcal{P}^n) \quad & = -2(\delta t)\langle\nabla m^n, \nabla\varphi\rangle + \langle(Id - m^n \times (m^n \times \Delta W^n), \varphi\rangle \\ & + \tfrac{1}{2}(\delta t)\textstyle\sum_k\langle(Id - m^n\times)(m^n \times G_k) \times G_k, \varphi\rangle \end{aligned}$$

- Set

$$m^{n+1}_{\delta t}(x) = \frac{m^n_{\delta t}(x) + v^n_{\delta t}(x)}{|m^n_{\delta t}(x) + v^n_{\delta t}(x)|}.$$

Note that $(\mathcal{P}^n)$ is the variational formulation of a fully implicit discretization of the Itô form of (SLLG).

The existence an uniqueness of the solution of $(\mathcal{P}^n)$ follows from classical theorem.

Theorem (F.A, AdB, A.H, 2014) :

Let $T > 0$ fixed, and $\delta t = T/N$; set $m_N(t) = m^n_{\delta t}$ for $t \in [n\delta t, (n+1)\delta t)$; then, up to a subsequence, the sequence $(m_N)_{N \in \mathbf{N}}$ converges in law in $L^2((0,T) \times \mathcal{O})$ to a martingale solution of (SLL).

Moreover, this solution satisfies (SLLG) and there is actually convergence in $L^2(\tilde{\Omega} \times (0,T) \times \mathcal{O})$, where $\tilde{\Omega}$ is the Skohorod space.

Some remarks :

- Fully discrete version easily deduced : replace $\mathbf{W}_{n,\delta t}$ by a finite element space of functions $\perp m^n_{\delta t}$
- Could use more general θ-scheme $\rightsquigarrow$ convergence for $\theta > 1/2$
- The scheme is linearly implicit (but nonlinearity hidden in the space $\mathbf{W}_{n,\delta t}$)
- Some 1-D version (i.e. for one dimensional noise) of the projected scheme was studied previously (Goldys-Le-Tran, 2013), but uses Doss-Sussman formulation $\rightsquigarrow$ restricted to 1-D noise
- As a by-product we obtain the equivalence of the two formulations of the equation (SLL) and (SLLG) in some sense
- Nothing known about convergence of the term $m^n|\nabla m^n|^2$

More on weak solutions

$$du = (\Delta u + u|\nabla u|^2 + u \times \Delta u + F_\phi(u))dt + u \times dW_\phi$$

on a domain $D \subset \mathbf{R}^2$ (typically torus in 2D) ; $u(t,x) \in \mathbf{R}^3$.

Global existence of finite energy solutions :

For $u_0 \in \mathbf{H}^1$, with $|u_0(t,x)| = 1$, a.s., there exists a global solution and a sequence of (random) stopping times $T^1 < T^2 < \cdots < T^k$ such that $\lim_{k\to\infty} T^k = +\infty$, a.s. and

$$u \in \cup_{k\in\mathbf{N}} C([T^k, T^{k+1}); \mathbf{H}^1) \cap L^2([T^k, T^{k+1}); \mathbf{H}^2).$$

Moreover,

$$u(t) \rightharpoonup u(T^k) \text{ as } t \nearrow T^k, \text{ i.e. } u \in C_w(\mathbf{R}^+; H^1).$$

Note that T^k characterized by "bubbles" :

$$\inf_{R>0} \sup_{\substack{x \in D \\ t \in [T^{k-1}, T^k)}} \int_{B(x,R)} |\nabla u(t,y)|^2 dy > \varepsilon_1$$

Here ε_1 is a fixed constant that measures the energy loss at the blow up points.

Remarks :

- Blow up at finite number of (space-time) points : the energy loss is quantified
- Deterministic case : Struwe, 1985
- Stochastic case : no energy conservation, but refined analysis of energy density evolution allows to assert that bubbling does not occur "too often" + compactness

Moreover

- Any weak solution is of this form (hence unique) provided $\frac{1}{2}|\nabla u|^2_{L^2} - c_\phi t$ is a supermartingale (deterministic case : Freire, 1995)

Example : $D = D(0,1) \subset \mathbf{R}^2$

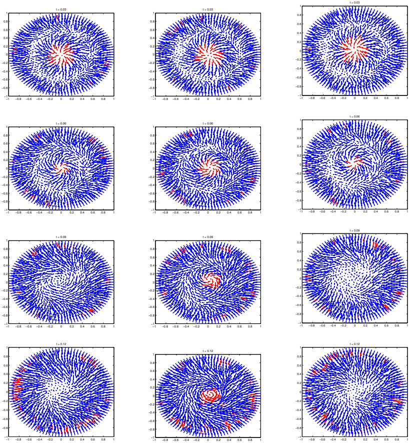


Figure 6.3 – View from above of 3 trajectories (the three columns) solution at times $t = 0$, $t = 0.015$, $t = 0.05$, $t = 0.06$ (raws). The color red means that $u^3(\omega, t, x) > 0$, whereas blue means $u^3(\omega, t, x) \leq 0$. The parameters are $\gamma = 0$, $k = 0.001$, $h_{\min} = 0.050518$. All solutions start with the same initial data, see figure 6.2.

Stochastic heat flow of harmonic maps :

$$du = (\Delta u + u|\nabla u|^2 + F_\phi(u))dt + u \times dW_\phi$$

The equivariant case :

Here D is the unit disc of $\mathbf{R}^2$; $x = re^{i\theta}$, and

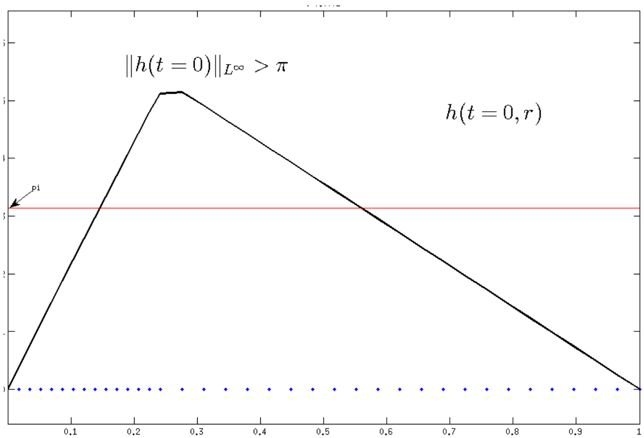
$$u(t,x) = (\cos\theta \sin h(t,r), \sin\theta \sin h(t,r), \cos h(t,r))$$

with noise $u^\perp \dot{W} = u \times e_\theta \dot{W}$ and W real valued, then

$$\begin{cases} dh = (\partial_{rr} h + \frac{1}{r}\partial_r h - \frac{\sin 2h}{2r^2})dt + dW \\ h(t,0) = h(t,1) = 0 \\ h(0,r) = h_0(r) \end{cases}$$

Blowup in the deterministic case may occur :
Chang, Ding, Yue, Ye, 1992

Blow up in the deterministic case :



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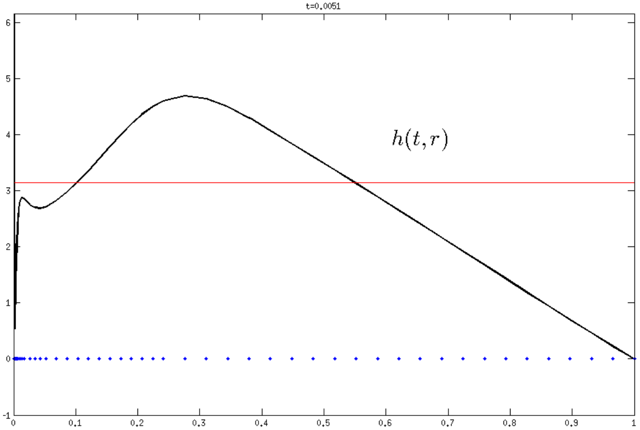

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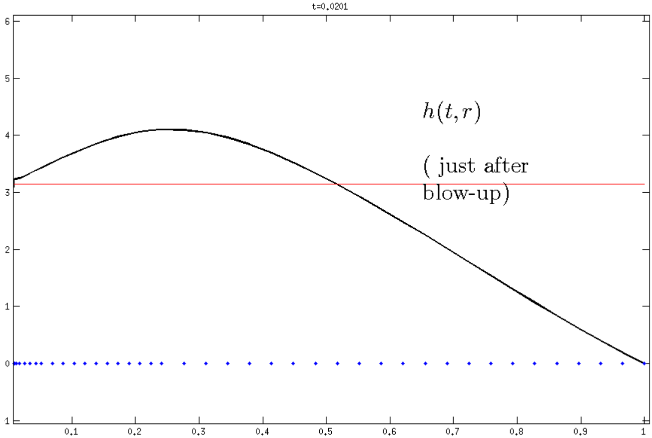
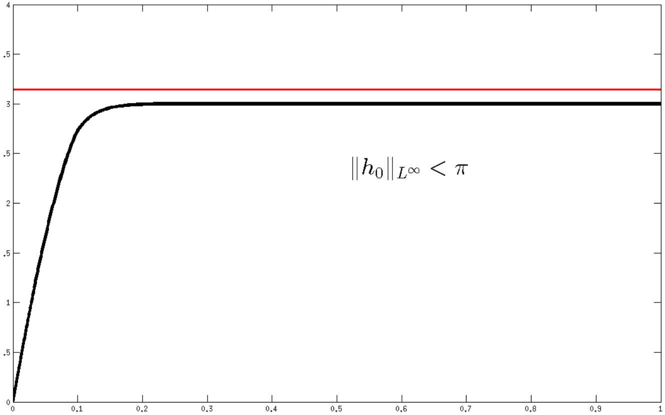
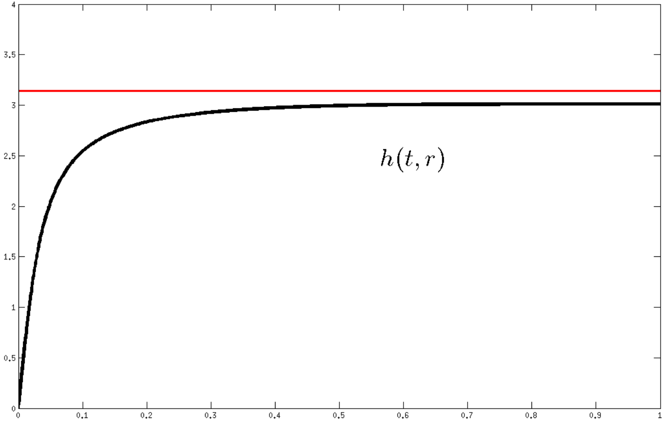


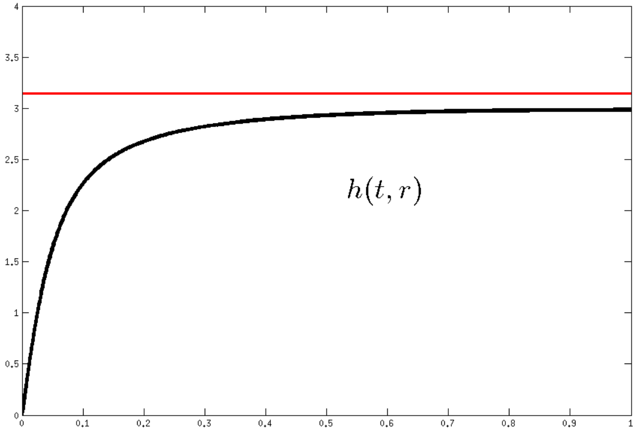
No blow up in the deterministic case :



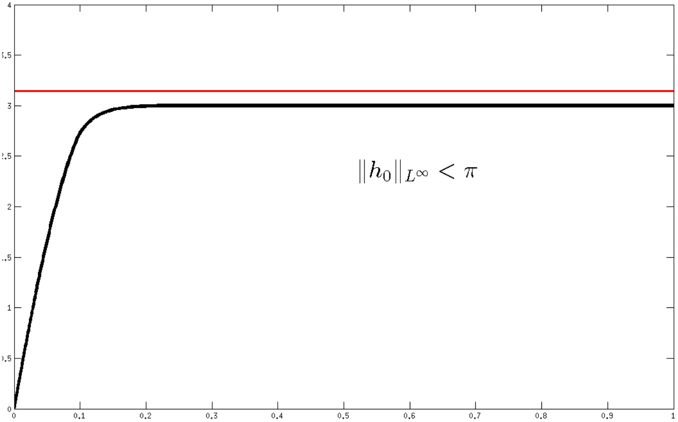
No blow up in the deterministic case :



No blow up in the deterministic case :



The stochastic case :



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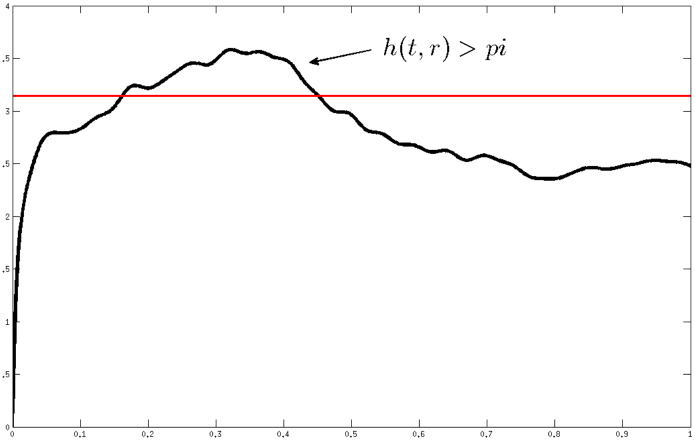
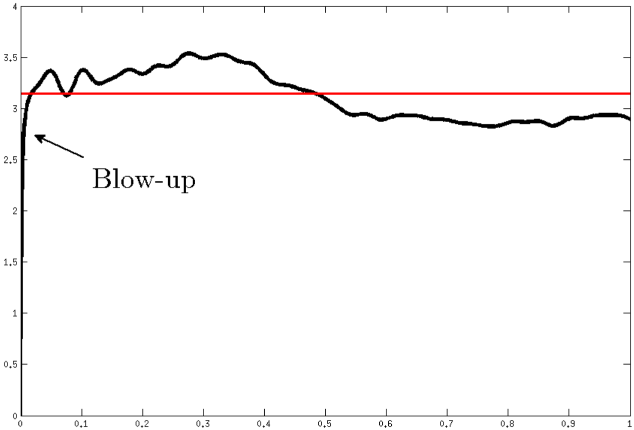


The stochastic case :



Let $A\varphi = \partial_{rr}\varphi + (\frac{1}{r}\partial_r - \frac{1}{r^2})\varphi$, with Dirichlet B.C. and let $V_\beta = D(A^\beta)$; then

Theorem

- For any β with $4/3 < \beta$ and any $h_0 \in V_\beta$, there is a unique strong local solution h of the equation in $C([0, \tau^*_\beta(h_0)), V_\beta)$
- If the noise is non degenerate, i.e. $\mathrm{Ker}(\phi^*) = \{0\}$, then for all $h_0 \in V_\beta$ and all $t^* > 0$,

$$\mathbf{P}(\tau^*_\beta(h_0) < t^*) > 0 \text{ and } \mathbf{P}(\sup_{[0,t^*)} \|\nabla u(t)\|_{L^\infty} = +\infty) > 0$$

Remarks

- Open for the LLG equation (deterministic or stochastic)
- Blow up for the Schrödinger maps : Bejenaru-Tataru 2010, Merle-Raphael-Rodianski 2011, Perelman 2012