Long term dynamics for nonlinear dispersive equations

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Linear asymptotic completeness

Linear Schrödinger equation in $\mathbb{R}^n$ with suitable decaying potential

$$i\partial_t \psi - \Delta\psi + V\psi = 0, \quad \psi(0) \in L^2(\mathbb{R}^d)$$

exhibits long-term dynamics

$$\psi(t) = \sum_j e^{itE_j}\psi_j + e^{-it\Delta}\phi_0 + o_{L^2}(1), \qquad t \to \infty$$

where $(-\Delta + V)\psi_j = E_j\psi_j$, $E_j \le 0$ are bound states, $\phi_0 \in L^2$.

Asymptotic completeness of the wave operators

Analogue for nonlinear equation? Soliton resolution problem.

Cubic nonlinear Klein-Gordon

Energy subcritical model equation:

$$\Box u + u = u^3 \ \text{ in } \ \mathbb{R}^{1+3}_{t,x}$$

$\forall\ \vec{u}(0) \in \mathcal{H} := H^1 \times L^2$, there $\exists!$ strong solution (Duhamel sense)

$$u \in C^0([0,T); H^1),\ \dot{u} \in C^0([0,T); L^2)$$

for some $T \geq T_0(\|\vec{u}[0]\|_{\mathcal{H}}) > 0$.

Properties: continuous dependence on data; persistence of regularity; energy conservation:

$$E(u,\dot{u}) = \int_{\mathbb{R}^3} \Big(\frac{1}{2}|\dot{u}|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 - \frac{1}{4}|u|^4\Big)\, dx$$

If $\|\vec{u}(0)\|_{\mathcal{H}} \ll 1$, then global existence; let $T^* > 0$ be maximal forward time of existence:

$$T^* < \infty \Longrightarrow \|u\|_{L^3([0,T^*),L^6(\mathbb{R}^3))} = \infty$$

Basic well-posedness, focusing cubic NLKG in $\mathbb{R}^3$

If $T^* = \infty$ and $\|u\|_{L^3([0,T^*),L^6(\mathbb{R}^3))} < \infty$, then u scatters: $\exists\,(\tilde{u}_0, \tilde{u}_1) \in \mathcal{H}$ s.t. for $v(t) = S_0(t)(\tilde{u}_0, \tilde{u}_1)$ one has

$$(u(t), \dot{u}(t)) = (v(t), \dot{v}(t)) + o_{\mathcal{H}}(1) \quad t \to \infty$$

where $S_0(t)$ is the free KG evolution. If u scatters, then $\|u\|_{L^3([0,\infty),L^6(\mathbb{R}^3))} < \infty$.

Finite propagation speed: if $\vec{u}(0) = 0$ on $\{|x - x_0| < R\}$, then $u(t,x) = 0$ on $\{|x - x_0| < R - t, 0 < t < \min(T^*, R)\}$.

Finite time blowup: $T > 0$, exact solution to cubic NLKG

$$\varphi_T(t) \sim \sqrt{2}(T-t)^{-1} \qquad \text{as} \;\; t \to T,$$

Use finite propagation speed to cut off smoothly to neighborhood of cone $|x| < T - t$. Gives smooth solution to NLKG, blows up at $t = T$ or before.

Ground state, Payne-Sattinger theorem

Small data: global existence and scattering.

Large data: can have finite time blowup.

Is there a criterion to decide finite time blowup/global existence?

YES if energy is smaller than the energy of the **ground state** Q unique positive, radial solution of :

$$-\Delta\varphi + \varphi = \varphi^3, \quad \varphi \in H^1(\mathbb{R}^3) \tag{1}$$

Minimization problem

$$\inf\left\{\|\varphi\|_{H^1}^2 \mid \varphi \in H^1,\ \|\varphi\|_4 = 1\right\}$$

has radial solution $\varphi_\infty > 0$, decays exponentially,

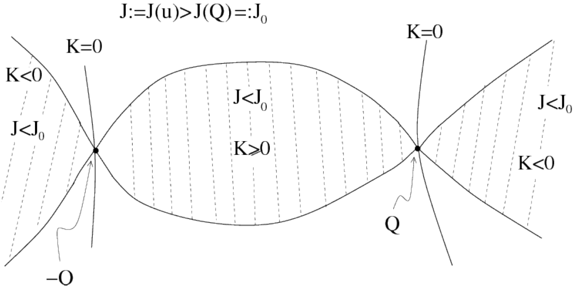
$Q = \lambda\varphi_\infty,\ \lambda > 0$. Minimizes the stationary energy (or action)

$$J(\varphi) := \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}|\varphi|^2 - \frac{1}{4}|\varphi|^4\right) dx = E(\varphi, 0)$$

amongst all nonzero solutions of (1). Dilation functional:

$$K_0(\varphi) = \langle J'(\varphi)|\varphi\rangle = \int_{\mathbb{R}^3} (|\nabla\varphi|^2 + |\varphi|^2 - |\varphi|^4)(x)\, dx$$

Payne-Sattinger theorem



$$J(Q) = \inf\{J(\varphi) \mid \varphi \in H^1 \setminus \{0\}, K_0(\varphi) = 0\}$$

Theorem (PS 1975)

If $E(u_0, u_1) < E(Q, 0)$, the dichotomy: $K_0(u_0) \geq 0$ global existence, $K_0(u_0) < 0$ finite time blowup

Ibrahim-Masmoudi-Nakanishi (2010): **Scattering** in addition to global existence. Why wait 35 years? See next slides...

Concentration Compactness by Bahouri-Gérard

Let $\{u_n\}_{n=1}^{\infty}$ free Klein-Gordon solutions in $\mathbb{R}^3$ s.t.

$$\sup_n \|\vec{u}_n\|_{L_t^\infty \mathcal{H}} < \infty$$

$\exists$ free solutions v^j bounded in $\mathcal{H}$, and $(t_n^j, x_n^j) \in \mathbb{R} \times \mathbb{R}^3$ s.t.

$$u_n(t,x) = \sum_{1 \le j < J} v^j(t + t_n^j, x + x_n^j) + w_n^J(t,x)$$

satisfies $\forall\ j < J$, $\vec{w}_n^J(-t_n^j, -x_n^j) \rightharpoonup 0$ in $\mathcal{H}$ as $n \to \infty$, and

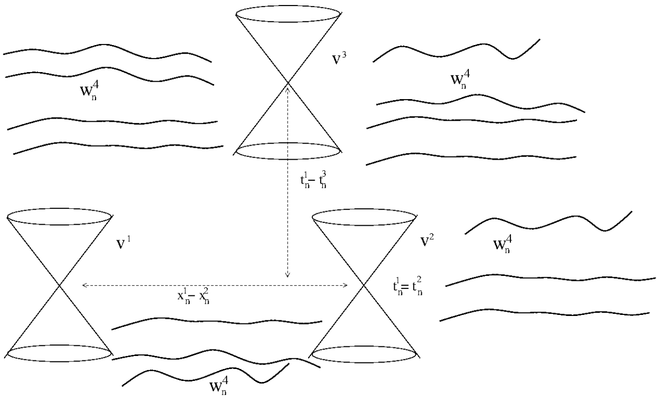
- $\lim_{n\to\infty}(|t_n^j - t_n^k| + |x_n^j - x_n^k|) = \infty \; \forall\, j \neq k$
- dispersive errors w_n^J vanish asymptotically:

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|w_n^J\|_{(L_t^\infty L_x^p \cap L_t^3 L_x^6)(\mathbb{R}\times\mathbb{R}^3)} = 0 \quad \forall\, 2 < p < 6$$

- orthogonality of the energy:

$$\|\vec{u}_n\|_{\mathcal{H}}^2 = \sum_{1 \le j < J} \|\vec{v}^j\|_{\mathcal{H}}^2 + \|\vec{w}_n^J\|_{\mathcal{H}}^2 + o(1) \quad n \to \infty$$

Profiles and Strichartz sea



We can extract further profiles from the Strichartz sea if w_n^4 does not vanish as $n \to \infty$ in a suitable sense. In the radial case this means $\lim_{n\to\infty} \|w_n^4\|_{L_t^\infty L_x^p(\mathbb{R}^3)} > 0$.

Kenig-Merle

Payne-Sattinger regime for the energy critical focusing NLW in $\mathbb{R}^3$:

$$u_{tt} - \Delta u - u^5 = 0$$

Stationary solution $W(x) = (1 + |x|^2/3)^{-\frac{1}{2}}$, unique radial solution. Aubin-Talenti solution, **extremizer** for the critical embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$.

Theorem (KM2007)

Assume $(u_0, u_1) \in \dot{H}^1 \times L^2$, $E(u_0, u_1) < E(W, 0)$.

- *If* $\|\nabla u_0\|_2 < \|\nabla W\|_2$ *then global existence and scattering (both time directions)*
- *If* $\|\nabla u_0\|_2 > \|\nabla W\|_2$ *then finite time blowup (both time directions). type I blowup, based on later work*

Kenig-Merle blueprint for scattering

- Small data scattering. Perturbative, based on Strichartz estimates.
- Induction on energy (Bourgain). Suppose result fails at some energy $0 < E_* < E(W,0)$. Use Bahouri-Gérard decomposition to find special solution u_* of energy E_*, with infinite scattering norm $\|u_*\|_{L^8_{0<t<T^*,x}} = \infty$. It follows that trajectory (up to time of existence T^*) is precompact, modulo scaling symmetry. Main point in concentration-compactness: there can be only one profile, and dispersive error vanishes in energy norm.
- Rigidity step: Show there can be no precompact solution of energy below ground state energy other than zero. Key role played by monotone quantities such as virial or Morawetz which express asymptotic outgoing property of waves. virial: $\langle u_t, x \cdot \nabla u \rangle$. Spatial cutoffs needed. Alternative tool: Exterior energy estimates.

Acta 2008 Kenig-Merle paper more complicated, exclusion of self-similar blowup, self-similar coordinates.

Beyond Payne Sattinger in unstable case

Theorem (Nakanishi-S. 2010)

Let $E(u_0, u_1) < E(Q, 0) + \varepsilon^2$, $(u_0, u_1) \in \mathcal{H}_{\mathrm{rad}}$. *In* $t \geq 0$ *for NLKG:*

1. *finite time blowup*
2. *global existence and scattering to* 0
3. *global existence and scattering to* Q*:*
   $u(t) = Q + v(t) + o_{H^1}(1)$ *as* $t \to \infty$, *and*
   $\dot{u}(t) = \dot{v}(t) + o_{L^2}(1)$ *as* $t \to \infty$, $\Box v + v = 0$, $(v, \dot{v}) \in \mathcal{H}$.

All 9 combinations of this trichotomy allowed as $t \to \pm\infty$.

- Applies to all dimensions, subcritical equations for which small data scattering is known.
- Linearized operator $-\Delta + 1 - 3Q^2$ has unique negative eigenvalue. crucial!
- third alternative is center-stable manifold of codimension 1 .

The invariant manifolds

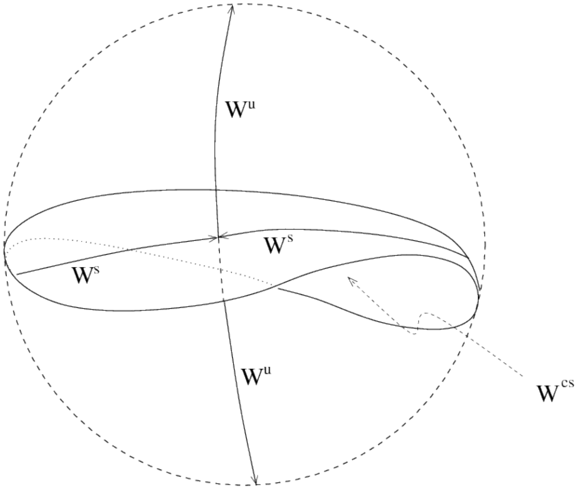
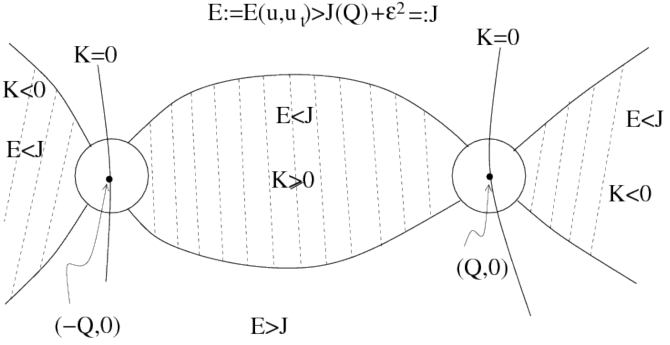


Figure: Stable, unstable, center-stable manifolds

Variational structure above $E(Q, 0)$



- Solution can pass through the balls. Energy is no obstruction anymore as in the Payne-Sattinger case.
- Key to description of the dynamics: One-pass (no return) theorem. The trajectory can make only one pass through the balls.
- Point: Stabilization of the sign of $K(u(t))$.

Numerical 2-dim section through $\partial\mathcal{S}_+$ (with R. Donninger)

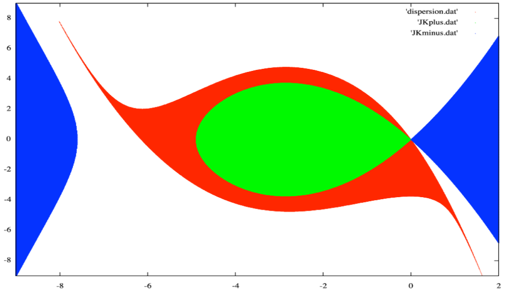


Figure: $(Q + Ae^{-r^2}, Be^{-r^2})$

- soliton at $(A, B) = (0, 0)$, (A, B) vary in $[-9, 2] \times [-9, 9]$
- RED: global existence, WHITE: finite time blowup, GREEN: $\mathcal{PS}_+$, BLUE: $\mathcal{PS}_-$
- Our results apply to a neighborhood of $(Q, 0)$, boundary of the red region looks smooth (caution!)

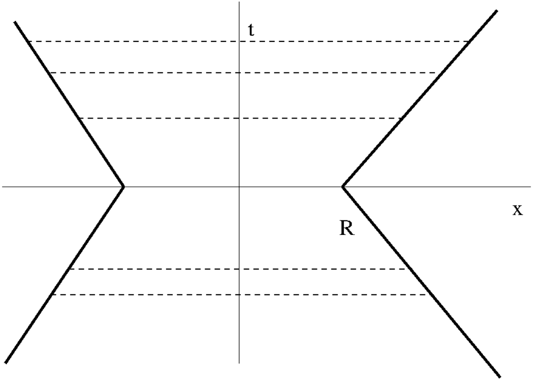
Center stable manifold for critical u^5 NLW

Theorem of Nakanishi-S is in subcritical radial regime (nonradial analogue exists, too). Analogue in the original Kenig-Merle critical setting?

Additional scaling invariance: any center-stable manifold contains curve $W_\lambda = \sqrt{\lambda} W(\lambda \cdot)$, $0 < \lambda < \infty$ of solitons. More importantly, it also contains blowup solutions with energy slightly above that of W.

Such blowup solutions were constructed by **Krieger-S-Tataru** in finite time, and by **Donninger-Krieger** in infinite time (more generally: non scattering solution in infinite time).

Nakanishi-Krieger-S 2011, 2013: existence of a center-stable manifold which separates global existence from blowup.
Open problems: Show blowup off the manifold is Type I. More information about dynamics on the manifold. What if **energy is much larger than ground state energy?**



$\mathbb{R}^3$ radial data, free wave $\Box u = 0$. Then ($R = 0$ case!) for one sign $\pm$

$$\lim_{t \to \pm\infty} \int_{|x| \geq |t|} (|\nabla u|^2 + u_t^2)(t, x)\, dx \geq c \int_{\mathbb{R}^3} (|\nabla u|^2 + u_t^2)(0, x)\, dx$$

Exterior Energy Estimates

Extends to all odd dimensions, nonradial data. Fails in even dimensions, but holds for data $(u_0, 0)$, $d = 4, 8, \ldots$, or $(0, u_1)$, $d = 6, 10, \ldots$ (**Côte, Kenig. S.**)

Obstruction for the case $R > 0$: Newton potential $u(x) = |x|^{-1}$ solves $\Box u = 0$ in $|x| > |t|$, has finite energy on $|x| \geq R > 0$ but infinite energy on $\mathbb{R}^3$.

If $u_0 \perp |x|^{-1}$ in $\dot{H}^1(|x| \geq R)$ radial, then

$$\begin{aligned} \lim_{t \to \pm\infty} \int_{|x| \geq |t|+R} (|\nabla u|^2 + u_t^2)(t,x)\, dx &\geq c \int_{|x| \geq R} (|\nabla u|^2 + u_t^2)(0,x)\, dx \\ &= c \int_R^\infty ((ru)_r^2 + (ru)_t^2)(0,r)\, dr \end{aligned}$$

Analogue in higher odd dimensions but with more obstructions (**Lawrie, Liu, Kenig, S.**).

Exterior Energy Estimates, nonlinear context

Theorem (DKM2012)

Let (u, u_t), radial finite energy solution of $\Box u - u^5 = 0$, $0 \le t < T^*$. If $u \neq 0, W_\lambda, \forall\, \lambda > 0$, then $\exists\, R > 0, \eta > 0$

$$\int_{|x| \ge |t|+R} (|\nabla u|^2 + u_t^2)(t,x)\,dx \ge \eta, \qquad 0 \le \pm t < T^*$$

- In particular, nonstationary global solutions radiate off a positive amount of energy.
- Find sequence $t_n \to \infty$ so that $\vec{u}(t_n)$ bounded in $\dot{H}^1 \times L^2$. Apply concentration compactness to $\vec{u}(t_n) - \vec{u}_L(t_n)$ where u_L is a free wave which carries all energy of $\vec{u}$ in $|x| \ge t - A$.
- Use theorem to identify all nonzero profiles as W_λ, and to prove radiative error vanishes.

DKM soliton resolution

Theorem (DKM2012)

Let (u, u_t), radial finite energy solution of $\Box u - u^5 = 0$, $0 \le t < T^*$.

- *Type I finite time blowup ($\dot{H}^1 \times L^2$ norm becomes infinite).*
- *Type II finite time blowup, multi-bubble representation via W_λ plus a function constant in time.*
- *Global bounded solutions, multi-bubble representation via W_λ plus free radiation.*

Multi-bubble in infinite time: exists free wave $\vec{v}$ s.t.

$$\vec{u}(t) = \sum_{j=1}^{J} (\pm W_{\lambda_j(t)}(t), 0) + (v(t), v_t(t)) + o(1)$$

$$\lambda_1(t) \ll \lambda_2(t) \ll \cdots \ll \lambda_J(t) \ll t, \qquad t \to \infty$$

In finite time, replace $\vec{v}$ by a constant. Absence of self-similar solutions.

DKM soliton resolution

Existence of such solutions known for one bubble: **Krieger-S-Tataru** for finite time, **Donninger-Krieger** in infinite time. One expects multi-bubble solutions to be unstable.

DKM method applied to other scenarios:

- Exterior equivariant wave maps $u : \mathbb{R}^3 \setminus B(0,1) \to \mathbb{S}^3$ with Dirichlet condition on ∂B and arbitrary data of finite energy. Scatter to the unique harmonic map in the same equivariance and degree class as the data. **Lawrie-S 11** for zero degree and 1-equivariance, **Kenig-Lawrie-S 13** for nonzero degree, **Kenig-Lawrie-Liu-S 14** for all equivariance classes and degrees.

  Observed numerically by **Bizon-Cmaj-Maliborski**.
- Defocusing (and thus stable) radial u^5 NLW in $\mathbb{R}^3$ with a potential well. The latter combines exterior energy estimates with center-stable manifolds and one-pass theorem (**Jia, Liu, S, Xu**).
- **Method appears not to apply in the subcritical case (propagation speed of Klein-Gordon).**

Defocusing u^5 NLW with potential

Consider

$$\Box u + Vu + u^5 = 0$$

radial, decaying V, deep enough to trap bound states $-\Delta\varphi + V\varphi + \varphi^5 = 0$. For generic V finitely many bound states, and linearized operator $H_\varphi := -\Delta + V + 5\varphi^4$ has no anomalies (zero energy resonance or eigenvalues). $\dot{H}^1 \times L^2(\mathbb{R}^3)$ data lead to global solutions (standard). *Long term dynamics?*

Theorem (Jia, Liu, S, Xu '14, '15)

All radial finite energy solutions scatter (asymptotically free) to one of the stationary solutions φ. Data scattering to φ are (i) open if H_φ has no negative eigenvalues (ii) form a C^1 path-connected manifold $\mathcal{M}$ in $\dot{H}^1 \times L^2(\mathbb{R}^3)$ of co-dimension equal to number of negative eigenvalues of H_φ.

The manifold $\mathcal{M}_\varphi$ is a global, unbounded, center-stable manifold associated with stationary solution φ. Is it closed?

Defocusing u^5 NLW with potential

- Scattering result is an adaptation of DKM technique. One profile in Bahouri-Gérard decomposition sees potential V (no scaling) the others do not (scaling).
- Potential V perturbative error in $|x| \geq t - A$, so exterior energy methods still apply.
- Local construction of $\mathcal{M}_\varphi$ near any solution scattering to φ. Delicate, radial endpoint for Strichartz. Note difference from standard center-stable manifold constructions: not near stationary solution but near a given scattering solution.
- The local manifold has repulsive property: If solution remains near it for all times $t \geq 0$, then it lies on it. Perturbative.
- Solution leaves, comes back eventually? Nonperturbative.
- **No-return** or **one-pass theorem**: if the solution exits small neighborhood of $\mathcal{M}_\varphi$ then it must emit a fixed quantum of energy which pushes it away from $\mathcal{M}_\varphi$, precluding a near return. So near but off of $\mathcal{M}_\varphi$ solution cannot scatter to φ.

The concentration-compactness approach to scattering also applies in absence of Payne-Sattinger dichotomy: defocusing, stable situations. *Wave maps* $\Phi : (\mathcal{M}, \boldsymbol{\eta}) \to (\mathcal{N}, \mathbf{g})$ are critical points of the Lagrangian

$$\mathcal{S}[\Phi] = \int \boldsymbol{\eta}^{\mu\nu} \langle (\mathrm{d}\Phi)_\mu, (\mathrm{d}\Phi)_\nu \rangle_{\mathbf{g}} \, \mathrm{dVol}_{\boldsymbol{\eta}}$$

In local coordinates

$$\Box_{\mathcal{M}} \Phi^k = \boldsymbol{\eta}^{\alpha\beta} \Gamma^k_{ij}(\Phi) \partial_\alpha \Phi^i \partial_\beta \Phi^j$$

Set $\mathcal{M} := \mathbb{R} \times M$, where M Riemannian, metric $\mathbf{h}$. Often h standard Euclidean metric.

Coercive conserved energy

$$\mathcal{E}[(\Phi, \dot{\Phi})](t) = \frac{1}{2} \int_{\{t\} \times M} \langle \dot{\Phi}, \dot{\Phi} \rangle_{\mathbf{g}}(t) + \mathbf{h}^{ij} \langle (\mathrm{d}\Phi)_i, (\mathrm{d}\Phi)_j \rangle_{\mathbf{g}}(t) \, \mathrm{dVol}_{\mathbf{h}} \,.$$

Wave maps from $\mathbb{R}^{1+2}$

Energy critical case $\mathcal{M} = \mathbb{R}^{1+2}$, scaling of equation identical to that of the energy.

Theorem

Consider the Cauchy problem (CP) for the wave map equation in the case $\mathcal{M} = \mathbb{R}^{1+2}$ with initial data of energy $E < \infty$.

1. *If $(\mathcal{N}, \mathbf{g})$ is negatively curved (i.e., all sectional curvatures < 0), then the (CP) is globally well-posed and the solution scatters to a constant map.*
2. *In general, global well-posedness of the (CP) and scattering to a constant map holds if $E < \mathcal{E}[(Q, 0)]$, where Q is the lowest energy non-trivial harmonic map $\mathbb{R}^2 \to \mathcal{N}$.*
3. *There exists a solution which blows up in finite time in the case $(\mathcal{N}, \mathbf{g}) = (\mathbb{S}^2, \mathbf{g}_{\mathbb{S}^2})$ and $E > \mathcal{E}[(Q, 0)]$.*

Klainerman, Machedon, Selberg, Tataru, Tao, Rodnianski, Nahmod, Stefanov, Uhlenbeck, Shatah, Struwe, Krieger, Sterbenz, Tataru

Role of harmonic maps

Consider the case $\mathcal{M} = \mathbb{R}^{1+2}$ and as an example $\mathcal{N} = \mathbb{S}^2$. Then part 2 is referred to as the Threshold theorem. It's a consequence of the following result.

Theorem (Struwe/ Sterbenz-Tataru bubbling)

If $\Phi(t)$ blows up at $t = 1$, then $\exists$ seq. $t_n \to 1$, $\lambda_n = o(1 - t_n)$ and $x_n \in \mathbb{R}^2$ so that the rescaled and translated seq.

$$\Phi_n(t,x) := \Phi(t_n + \lambda_n t, x_n + \lambda_n x) \to Q_\ell(t,x) \quad H^1_{loc}((-1,1) \times \mathbb{R}^2)$$

where Q_ℓ is a Lorentz transformed harmonic map from $\mathbb{R}^2 \to \mathcal{N}^2$.

- Similar result if $T_+ = +\infty$. Scattering to constant map or local convergence (up to symmetries) to harmonic map.
- However, the $E < \mathcal{E}[Q, 0]$ threshold result can be refined if one takes into account an additional invariant of the equation, namely the topological degree of the wave map.

Role of harmonic maps, topological invariants

Finite energy wave maps $I \times \mathbb{R}^2 \to \mathbb{S}^2$ have an integer valued topological degree, which is fixed by the evolution.

$$\deg(\Phi) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \Phi^* \omega_{\mathbb{S}^2} \in \mathbb{Z}$$

$\mathcal{E}[(\Phi, \dot{\Phi})] \geq 4\pi \deg[\Phi]$ with equality $\Leftrightarrow$ Φ is a *harmonic map*.

- Fin. en. HM to $\mathbb{S}^2$ are classified: Conformal maps $\mathbb{C}_\infty \to \mathbb{C}_\infty$, rational functions.
- All degree k maps with $k \geq 1$ have energy $\geq$ lowest energy nontrivial harmonic map Q_1 given by stereographic projection. Hence $E < \mathcal{E}[Q_1, 0]$ gwp. and scattering result is meaningless for maps with degree ≥ 1.

Theorem (Refined Threshold result (Lawrie-Oh '15))

Any finite energy wave map $\Phi(t) : \mathbb{R}^2 \to \mathbb{S}^2$ *with* $\deg(\Phi) = 0$ *and* $E < 2\mathcal{E}[(Q_1, 0)]$ *is defined globally in time and scatters.*

Equivariant wave maps

Equivariant wave maps: characterization of blow-up/scattering dynamics for degree $k \geq 1$. $\Phi(t, r, \theta) = (\psi(t, r), \theta)$, WM eq. is

$$\psi_{tt} - \psi_{rr} - \frac{1}{r}\psi_r + \frac{\sin 2\psi}{2r^2} = 0$$

Theorem (Côte, Kenig, Lawrie, S. '12)

$\Phi = (\psi, \theta)$ *smooth equiv. WM,* $\deg(\Phi) = 1$ *with* $\mathcal{E}(\vec{\psi}) < 3\mathcal{E}(Q, 0)$

- *If* $T_+ < \infty$ *(say* $T_+ = 1$*) then,* $\exists \lambda : [0, 1) \to (0, \infty)$, $\lambda(t) = o(1 - t)$, *a map* $\vec{\varphi} = (\varphi_0, \varphi_1) \in \mathcal{H}_0$ *with* $\mathcal{E}(\vec{\varphi}) = \mathcal{E}(\vec{\psi}) - \mathcal{E}(Q)$, *and a decomposition*

$$\vec{\psi}(t) = \vec{\varphi} + \left(Q\left(\cdot/\lambda(t)\right), 0\right) + o_{\mathcal{H}_0}(1) \quad as \quad t \to 1$$

- *If* $T_+ = \infty$ *then* $\exists\ \lambda : [0, \infty) \to (0, \infty)$ *with* $\lambda(t) = o(t)$, *a solution to linearized equation* $\vec{\varphi}_L(t) \in \mathcal{H}_0$, *s.t.*

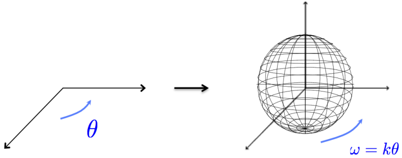
$$\vec{\psi}(t) = \vec{\varphi}_L(t) + \left(Q\left(\cdot/\lambda(t)\right), 0\right) + o_{\mathcal{H}_0}(1) \quad as \quad t \to \infty$$

Further results: **Côte '14, Jia-Kenig '15**

Equivariant wave maps from $\mathbb{R} \times \mathbb{H}^2$

Wave maps Φ from $\mathcal{M} = \mathbb{R} \times \mathbb{H}^2$ to $(\mathcal{N}, \mathbf{g})$

- $\mathcal{N} = \mathbb{H}^2$ or $\mathbb{S}^2$, with geod. polar coord. (ψ, ω), $ds^2 = d\psi^2 + g^2(\psi)d\omega^2$ where $g(\psi) = \sinh\psi$ when $\mathcal{N} = \mathbb{H}^2$ and $g(\psi) = \sin\psi$ when $\mathcal{N} = \mathbb{S}^2$.
- Φ is k-equivariant: $\Phi(t, r, \theta) = (\psi(t, r), k\theta)$



Equation and Energy:

$$\partial_t^2\psi - \frac{1}{\sinh r}\partial_r(\sinh r\,\partial_r\psi) + k^2\frac{g(\psi)g'(\psi)}{\sinh^2 r} = 0$$

$$\mathcal{E}[\psi, \partial_t\psi](t) = \frac{1}{2}\int_0^\infty \left((\partial_t\psi)^2 + (\partial_r\psi)^2 + \frac{k^2 g(\psi)^2}{\sinh^2 r}\right)\sinh r\,\mathrm{d}r$$

Theorems by Lawrie-Oh-Shahshahani

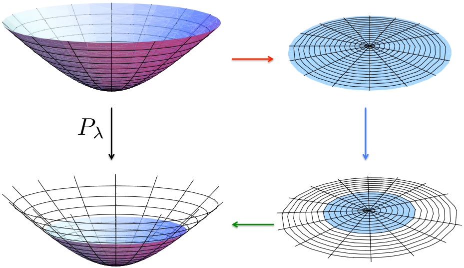
There exists family $P_\lambda : \mathbb{H}^2 \to \mathbb{H}^2$ finite energy harmonic maps, $0 \leq \lambda < 1$.

Theorem (LOS, '15)

Let $0 \leq \lambda < \Lambda$, where $\Lambda = 0.56\ldots$. (CP) for 1-equivariant WM $\mathbb{R} \times \mathbb{H}^2 \to \mathbb{H}^2$, with finite energy initial data (ψ_0, ψ_1), is globally well-posed and solution scatters to P_λ as $t \to \pm\infty$. Here $P_\lambda(\infty) = \psi_0(\infty)$.

No energy restriction, but spatial endpoint restriction. Excess energy moves off the spatial infinity. A key technical ingredient is a Bahouri-Gérard type profile decomposition established in a recent preprint, (LOS '14), following recent work of **Ionescu, Pausader, Staffilani** on the NLS.

The harmonic maps in LOS theorem



That these are all of the finite energy harmonic maps in the first equivariance class follows from an ODE argument.

Dispersive equations with dissipation

Consider in $\mathbb{R}^d$, $d \le 6$

$$\partial_{tt} u - \Delta u + 2\alpha \partial_t u + u - f(u) = 0$$

data $(u(0), \partial_t u(0)) \in H^1 \times L^2(\mathbb{R}^d)$, $\alpha > 0$, $f \in C^{1,\beta}(\mathbb{R})$, odd, $f'(0) = 0$, subcritical. Ambrosetti-Rabinowitz condition: there exists $\gamma > 0$ so that

$$\int_{\mathbb{R}^d} 2(1+\gamma)F(\varphi) - \varphi f(\varphi) \le 0 \quad \forall \varphi \in H^1(\mathbb{R}^d), \quad F' = f \qquad (\star)$$

For example

$$f(u) = \sum_{i=1}^{m_1} a_i |u|^{p_i - 1} u - \sum_{j=1}^{m_2} b_j |u|^{q_j - 1} u \ , \ 1 < q_j < p_i \le \frac{d+2}{d-2}, \forall i, j \qquad (\dagger)$$

$$a_i, b_j \ge 0, a_{m_1} > 0 \ .$$

For this class existence, uniqueness of ground state known, hyperbolicity of linearized operator. We only assume $(\star)$ not $(\dagger)$.

Convergence to equilibria or blowup

Theorem (Burq-Raugel-S '15)

Let $\alpha > 0$. Assume that $1 \leq d \leq 6$ and that nonlinearity satisfies above conditions. Then any finite energy solution

1. *either blows up in finite time,*
2. *or exists globally and converges to an equilibrium point (stationary solution) as $t \to +\infty$.*

Does not use concentration-compactness, but relies heavily of results from dynamical systems in infinite dimensions (invariant manifold theory, **Chen-Hale-Tan, Brunovsky-Polacik** 90s).

Energy is monotone decreasing:

$$\frac{d}{dt}E(\vec{u}(t)) = -2\alpha \int_{\mathbb{R}^d} u_t^2 \, dx$$

Implies: ω-limit set of any solution consists of equilibria (stationary solutions).

Convergence to equilibria or blowup: scheme of proof

- Not clear a priori that a global solution is bounded in $H^1 \times L^2$.
- Let $K_0(\varphi) = \int_{\mathbb{R}^d} |\nabla\varphi|^2 + \varphi^2 - \varphi f(\varphi)\,dx$. Show $\exists\, t_n \to \infty$ s.t. $K_0(t_n) \to 0$.
- Then show that $\vec{u}(t_n) \to (Q, 0)$, a stationary solution.
- Linearize about $(Q, 0)$. We may or may not have hyperbolicity of the linearize wave equation, depends on whether $H_Q := -\Delta + 1 - f'(Q)$ has trivial kernel or not; in latter case kernel is 1-dimensional (due to radial assumption).
- Construct stable, unstable, center manifolds near $(Q, 0)$. Latter only present if H_Q has a nontrivial kernel. If present, then center manifold is a curve.
- Now apply Brunovsky-Polacik: if center dynamics is stable, $\vec{u}(t) \not\to (Q, 0)$ as $t \to \infty$ implies $\vec{u}(\tilde{t}_n) \to (\tilde{Q}, 0) \neq (Q, 0)$ which belongs to unstable manifold. But such an equilibrium cannot lie on unstable manifold, so done. Stability of center manifold: it is a curve, and infinitely many equilibria on it. So evolution is trapped between them.

The spectrum of the linearized flow with dissipation