SPDEs on graphs as limit of SPDEs on narrow channels

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The equation in the domain G

Let G be a bounded domain in $\mathbb{R}^2$, having a smooth boundary ∂G. We consider here the following SPDE in G, endowed with co-normal boundary conditions,

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t}(t,x,y) = \mathcal{L}_\epsilon u(t,x,y) + b(u_\epsilon(t,x,y)) + \dfrac{\partial w^Q}{\partial t}(t,x,y), \\[2ex] \dfrac{\partial u_\epsilon}{\partial \nu_\epsilon}(t,x,y) = 0, \quad (x,y) \in \partial G, \qquad u_\epsilon(0,x,y) = u_0(x,y), \end{cases} \tag{1}$$

where, for every $\epsilon > 0$,

$$\mathcal{L}_\epsilon = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{1}{\epsilon^2}\frac{\partial^2}{\partial y^2}\right).$$

Here we assume:

- $\nu_\epsilon = \nu_\epsilon(x, y)$ is the unit interior conormal at ∂G, corresponding to the second order differential operator $\mathcal{L}_\epsilon$;

- $w^Q(t, x, y)$ is a cylindrical Wiener process in $L^2(G)$. That is

$$w^Q(t, x, y) = \sum_{k=1}^{\infty} (Qe_k)(x, y)\beta_k(t), \qquad t \geq 0, \qquad (x, y) \in G,$$

  where Q is a bounded linear operator on $L^2(G)$, $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis in $L^2(G)$ and $\{\beta_k(t)\}_{k\in\mathbb{N}}$ is a sequence of independent Brownian motions on some stochastic basis $(\boldsymbol{\Omega}, \mathcal{F}, \{\mathcal{F}_\mathbf{t}\}_{t\geq 0}, \mathbf{P})$.

- The function $b : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous;

- $u_0 \in C(\bar{G})$.

For any $T > 0$ and $p \geq 1$

equation (1) admits a unique mild solution

$u_\epsilon \in L^p(\mathbf{\Omega}; C([0,T], L^2(G)))$, for every fixed $\epsilon > 0$.

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This means that there exists a unique adapted process $u_\epsilon \in L^p(\mathbf{\Omega}; C([0, T]; L^2(G)))$ such that

$$u_\epsilon(t) = S_\epsilon(t)u_0 + \int_0^t S_\epsilon(t-s)b(u_\epsilon(s))\,ds + \int_0^t S_\epsilon(t-s)dw^Q(s),$$

where

$S_\epsilon(t)$ is the semigroup generated by the realization L_ϵ in $L^2(G)$ of the differential operator $\mathcal{L}_\epsilon$, endowed with co-normal boundary condition.

The equation in the narrow channel G_ϵ

After a change of variables, equation (1) can be obtained from the equation

$$\begin{cases} \dfrac{\partial v_\epsilon}{\partial t}(t,x,y) = \dfrac{1}{2}\,\Delta v_\epsilon(t,x,y) + b(v_\epsilon(t,x,y)) + \sqrt{\epsilon}\,\dfrac{\partial w^{Q_\epsilon}}{\partial t}(t,x,y), \\[2ex] \dfrac{\partial v_\epsilon}{\partial \hat{\nu}_\epsilon}(t,x,y) = 0, \quad (x,y) \in \partial G_\epsilon, \qquad v_\epsilon(0,x,y) = u_0(x, y/\epsilon), \end{cases} \tag{2}$$

where G_ϵ is the narrow domain

$$G_\epsilon = \{(x,y) \in \mathbb{R}^2 \,:\, (x, y/\epsilon) \in G\},$$

$\hat{\nu}_\epsilon(x,y)$ is the inward unit normal vector at ∂G_ϵ,
and $w^{Q_\epsilon}(t)$ is a suitable cylindrical Wiener process in $L^2(G_\epsilon)$.

Reaction-diffusion equations of the same type as (2), with or without additional noise, arise, for example, in

models for the motion of molecular motors.

Actually, one of the possible ways to model Brownian motors/ratchets is to describe them as particles traveling along a designated track, and the designated track along which the molecule/particle is traveling can be viewed as a tubular domain with many *wings* added to it.

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Actually, one of the possible ways to model Brownian motors/ratchets is to describe them as particles traveling along a designated track, and the designated track along which the molecule/particle is traveling can be viewed as a tubular domain with many *wings* added to it.

It is worth mentioning that Bonaccorsi, Marinelli and Giglio have studied stochastic FitzHugh-Nagumo equations on networks with impulsive noise.

The problem

We are here interested in

the limiting behavior, as $\epsilon \downarrow 0$, of the solution u_ϵ of equation (1) or, equivalently, of the solution v_ϵ of equation (2).

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We want to show that

the solution of the SPDE on G (or, equivalently, on G_ϵ) converges, as $\epsilon \downarrow 0$, to the solution of a suitable SPDE on the graph Γ that can be associated with the domain G.

The Neumann problem

Let us consider the problem

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t}(t,x,y) = \dfrac{1}{2}\left[\dfrac{\partial^2 u}{\partial x^2} + \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u}{\partial y^2}\right](t,x,y), \\[2ex] \dfrac{\partial u_\epsilon}{\partial \nu_\epsilon}(t,x,y) = 0, \quad (x,y) \in \partial G, \qquad u_\epsilon(0,x,y) = u_0(x,y). \end{cases}$$

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The solution $u_\epsilon(t,x,y)$ of the Cauchy problem above has a probabilistic representation. Namely

$$u_\epsilon(t,x,y) = \mathbb{E}_{(x,y)} u_0(X^\epsilon(t), Y^\epsilon(t)),$$

where $(X^\epsilon(t), Y^\epsilon(t))$ is the diffusion process governed by the operator $\mathcal{L}_\epsilon$ inside G and undergoing instantaneous reflections at ∂G, with respect to the co-normal associated with $\mathcal{L}_\epsilon$.

Since

$$\mathcal{L}_\epsilon = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{1}{\epsilon^2}\frac{\partial^2}{\partial y^2}\right),$$

the process $(X^\epsilon(t), Y^\epsilon(t))$ has a slow component $X^\epsilon(t)$ and a fast component $Y^\epsilon(t)$, if $0 < \epsilon << 1$.

In particular, while $X^\epsilon(t)$ remains near a point x, $Y^\epsilon(t)$ moves very fast over the connected component of the cross section $C(x) = \{(x, y) \in G\}$, undergoing reflection at the ends of the interval and having approximately the uniform distribution on it.

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In what follows we shall denote

$$S_\epsilon(t)\varphi(x, y) = \mathbb{E}_{(x,y)}\varphi(X^\epsilon(t), Y^\epsilon(t)),$$

for every $\varphi : G \to \mathbb{R}$, Borel and bounded, and for every $(x, y) \in G$.

The Skorohod problem

The process $Z_\epsilon(t) = (X_\epsilon(t), Y_\epsilon(t))$ is the solution of the problem

$$dZ^\epsilon(t) = \sqrt{\sigma_\epsilon}\, dB(t) + \sigma_\epsilon\, \nu(Z^\epsilon(t))\, d\phi^\epsilon(t), \qquad Z^\epsilon(0) = z, \qquad (3)$$

where

$$\sigma_\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon^{-2} \end{pmatrix}, \qquad z = (x, y) \in G. \qquad (4)$$

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Here

- $\nu(x, y)$ is the inward unit normal at ∂G;
- $B(t)$ is a 2-dimensional standard Brownian motion, defined on some stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$;
- $\phi^\epsilon(t)$ is the local time of the process $Z^\epsilon(t)$ on ∂G, that is an $\mathcal{F}_t$-adapted process, continuous with probability 1, non-decreasing and increasing only when $Z^\epsilon(t) \in \partial G$.

More precisely:

the random pair $(Z^\epsilon(t), \phi^\epsilon(t))$, $t \geq 0$, is a solution of problem (3)

if $Z^\epsilon(t)$ is a $\bar{G}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$ semi-martingale and $\phi^\epsilon(t)$ is a non-decreasing continuous process, such that

$$Z^\epsilon(t) = z + \sqrt{\sigma_\epsilon}\, B(t) + \int_0^t \sigma_\epsilon\, \nu(Z^\epsilon(s))\, d\phi^\epsilon(s),$$

and

$$\phi^\epsilon(t) = \int_0^t I_{\{Z^\epsilon(s)\in \partial G\}}\, d\phi^\epsilon(s).$$

The identification map Γ

In what follows, we shall assume that the region G has smooth boundary and satisfies the following properties.

I. There are only finitely many $x \in \mathbb{R}$ for which $\nu_2(x, y) = 0$, for some $(x, y) \in \partial G$.

II. For every $x \in \mathbb{R}$, the cross-section $C(x) = \{(x, y) \in G\}$ consists of a finite union of intervals. Namely, when $C(x) \neq \emptyset$, there exist $N(x) \in \mathbb{N}$ and intervals $C_1(x), \ldots, C_{N(x)}(x)$ such that

$$C(x) = \bigcup_{k=1}^{N(x)} C_k(x).$$

III. If $x \in \mathbb{R}$ is such that $\nu_2(x, y) \neq 0$, then for any $k = 1, \ldots, N(x)$ we have

$$\alpha_k(x) := |C_k(x)| > 0.$$

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Namely, if we identify the points of each connected component $C_k(x)$ of each cross section $C(x)$, we obtain a graph Γ, with a finite number of vertices O_i, corresponding to the connected components containing points $(x, y) \in \partial G$ such that $\nu_2(x, y) = 0$, and with a finite number of edges I_k, connecting the vertices.

On our graph there are two different types of vertices, exterior ones, that are connected to only one edge of the graph, and interior ones, that are connected to two or more edges.

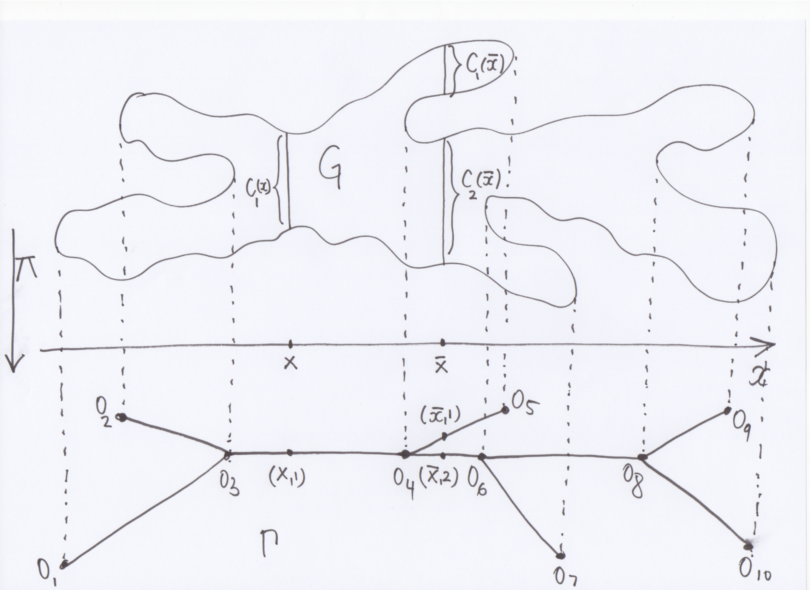

Figure : The identification map

A limiting result

Freidlin and Wentcell (PTRF 2012) have studied

the limiting behavior, as $\epsilon \downarrow 0$, of the (non Markov) process $\Pi_\epsilon(t) := \Pi(Z^\epsilon(t))$, $t \geq 0$, in the space $C([0, T]; \Gamma)$, for any fixed $T > 0$ and $z \in G$.

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They have shown that the process $\Pi_\epsilon(t)$, which describes the slow motion of the process $Z^\epsilon(t)$, converges, in the sense of weak convergence of distributions in the space of continuous Γ-valued functions, to a diffusion process $\bar{Z}$ on Γ.

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They have shown that the process $\Pi_\epsilon(t)$, which describes the slow motion of the process $Z^\epsilon(t)$, converges, in the sense of weak convergence of distributions in the space of continuous Γ-valued functions, to a diffusion process $\bar{Z}$ on Γ.

Namely, they have proved that for any bounded and continuous functional $F : C([0, T]; \Gamma) \to \mathbb{R}$ and $z \in G$ it holds

$$\lim_{\epsilon \to 0} \mathbb{E}_z F(\Pi_\epsilon(\cdot)) = \bar{\mathbb{E}}_{\Pi(z)} F(\bar{Z}(\cdot)).$$

The process $\bar{Z}$ has been described in terms of its generator $\bar{L}$, which is given by suitable differential operators $\bar{\mathcal{L}}_k$ within each edge $I_k = \{(x, k) : a_k \leq x \leq b_k\}$ of the graph and by certain gluing conditions at the vertices O_i of the graph.

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More precisely, for each k,

$$\bar{\mathcal{L}}_k f(x) = \frac{1}{2\alpha_k(x)} \frac{d}{dx}\left(\alpha_k \frac{df}{dx}\right)(x), \qquad a_k < x < b_k,$$

where $\alpha_k(x)$ is the length of the cross section $C_k(x)$.

The domain $D(\bar{L})$ is defined as the set of continuous functions on the graph Γ, that are twice continuously differentiable in the interior part of each edge of the graph, and satisfy suitable gluing conditions at the vertices .

In what follows, we shall denote by $\bar{S}(t)$, $t \geq 0$, the transition semigroup associated with $\bar{Z}(t)$, defined by

$$\bar{S}(t)f(x,k) = \bar{\mathbb{E}}_{(x,k)} f(\bar{Z}(t)), \qquad t \geq 0, \qquad (x,k) \in \Gamma,$$

for any Borel and bounded function $f : \Gamma \to \mathbb{R}$.

Back to the Neumann problem

Since the solution of the problem

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t}(t,x,y) = \dfrac{1}{2}\left[\dfrac{\partial^2 u}{\partial x^2} + \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u}{\partial y^2}\right](t,x,y), \\ \\ \dfrac{\partial u_\epsilon}{\partial \nu_\epsilon}(t,x,y) = 0, \quad (x,y) \in \partial G, \qquad u_\epsilon(0,x,y) = u_0(x,y), \end{cases}$$

is given by

$$u_\epsilon(t,x,y) = S_\epsilon(t)u_0(x,y) = \mathbb{E}_{(x,y)}u_0(Z^\epsilon(t)),$$

in order to study the asymptotics of u_ϵ one would like to use the limit

$$\lim_{\epsilon\to 0} \mathbb{E}_z F(\Pi_\epsilon(\cdot)) = \bar{\mathbb{E}}_{\Pi(z)} F(\bar{Z}(\cdot)).$$

Some notations

In what follows, we denote $H := L^2(G)$.

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Moreover, we denote by $\bar{H}$ the space of measurable functions $f : \Gamma \to \mathbb{R}$ such that

$$\sum_{k=1}^{N} \int_{I_k} |f(x,k)|^2 \alpha_k(x)\, dx < +\infty.$$

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$$\sum_{k=1}^{N} \int_{I_k} |f(x,k)|^2 \alpha_k(x)\, dx < +\infty.$$

The space $\bar{H}$ turns out to be a Hilbert space, endowed with the scalar product

$$\langle f, g \rangle_{\bar{H}} = \sum_{k=1}^{N} \int_{I_k} f(x,k) g(x,k) \alpha_k(x)\, dx =: \int_{\Gamma} f(z) g(z)\, \nu(dz).$$

Now, for any $u \in H$ we define

$$u^\wedge(x,k) = \frac{1}{\alpha_k(x)} \int_{C_k(x)} u(x,y)\,dy, \qquad (x,k) \in \Gamma,$$

and for any $f \in \bar{H}$ we define

$$f^\vee(x,y) = f(\Pi(x,y)), \quad (x,y) \in G.$$

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And, for any $f \in \bar{H}$, we have

$$(f^{\vee})^{\wedge} = f.$$

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and for any $f \in \bar{H}$ we define

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For any $f \in \bar{H}$ and $u \in H$ we have

$$\langle u^\wedge, f\rangle_{\bar{H}} = \langle u, f^\vee\rangle_H.$$

And, for any $f \in \bar{H}$, we have

$$(f^\vee)^\wedge = f.$$

But, in general, if $u \in H$

it is not true that $(u^\wedge)^\vee = u$.

The main approximation theorem

If the domain G satisfies the assumptions above, then

$$\lim_{\epsilon\to 0} \sup_{t\in[\tau,T]} \left|\mathbb{E}_z\,\varphi(Z^\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(z)}\,\varphi^\wedge(\bar{Z}(t))\right| = 0, \tag{5}$$

for any $\varphi \in C(\bar{G})$ and $z \in G$, and for any $0 \le \tau \le T$.

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for any $\varphi \in C(\bar{G})$ and $z \in G$, and for any $0 \le \tau \le T$.

This means that

$$\lim_{\epsilon\to 0} \sup_{t\in[\tau,T]} |S_\epsilon(t)\varphi(z) - \bar{S}(t)\varphi^\wedge(\Pi(z))| = 0.$$

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$$\lim_{\epsilon\to 0}\sup_{t\in[\tau,T]}\left|\mathbb{E}_z\,\varphi(Z^\epsilon(t))-\bar{\mathbb{E}}_{\Pi(z)}\,\varphi^\wedge(\bar{Z}(t))\right|=0, \tag{5}$$

for any $\varphi\in\ C(\bar{G})$ and $z\in\ G$, and for any $0\leq\tau\leq T$.

This means that

$$\lim_{\epsilon\to 0}\sup_{t\in[\tau,T]}|S_\epsilon(t)\varphi(z)-\bar{S}(t)\varphi^\wedge(\Pi(z))|=0.$$

Moreover, for any $\varphi\in\ H$

$$\lim_{\epsilon\to 0}\sup_{t\in[\tau,T]}|S_\epsilon(t)\varphi-(\bar{S}(t)\varphi^\wedge)^\vee|_H$$

$$=\lim_{\epsilon\to 0}\sup_{t\in[\tau,T]}|(S_\epsilon(t)\varphi)^\wedge-\bar{S}(t)\varphi^\wedge|_{\bar{H}}=0.$$

How to prove limit (5)

Limit (5) is not a straightforward consequence of the limit

$$\lim_{\epsilon \to 0} \mathbb{E}_z F(\Pi_\epsilon(\cdot)) = \bar{\mathbb{E}}_{\Pi(z)} F(\bar{Z}(\cdot)).$$

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Actually, (5) is a consequence of the following two limits

$$\lim_{\epsilon\to 0} \sup_{t\in[\tau,T]} \left|\mathbb{E}_z\, \varphi(Z^\epsilon(t)) - \mathbb{E}_z\, \varphi^\wedge(\Pi_\epsilon(t))\right| = 0, \tag{6}$$

and

$$\lim_{\epsilon\to 0} \sup_{t\in[\tau,T]} \left|\mathbb{E}_z \varphi^\wedge(\Pi_\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(z)} \varphi^\wedge(\bar{Z}(t))\right| = 0, \tag{7}$$

that have to be valid for any $0 < \tau < T$ and $z \in G$ and for any $\varphi \in C(\bar{G})$.

Limit (6)

Limit

$$\lim_{\epsilon \to 0} \sup_{t \in [\tau, T]} \left| \mathbb{E}_z \, \varphi(Z^\epsilon(t)) - \mathbb{E}_z \, \varphi^\wedge(\Pi_\epsilon(t)) \right| = 0,$$

follows from an averaging argument, and its proof requires a suitable localization in time in the same spirit of classical Khasminski's paper.

But here the localization procedure is more delicate than in the classical setting considered by Khasminski, as it involves a stochastic differential equation with reflection and hence requires suitable estimates for the time increments of the local time of the process $(X^\epsilon(t), Y^\epsilon(t))$ at the boundary of G.

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Moreover, the interior vertices require a special treatment.

An approximation result inside each edge

Assume

$$G = \left\{ (x,y) \in \mathbb{R}^2 \,:\, h_1(x) \leq y \leq h_2(x),\ x \in \mathbb{R} \right\},$$

for some functions $h_1, h_2 \in C_b^3(\mathbb{R})$, such that

$$h_2(x) - h_1(x) =: \alpha(x) \geq \alpha_0 > 0, \qquad x \in \mathbb{R}.$$

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$$h_2(x) - h_1(x) =: \alpha(x) \geq \alpha_0 > 0, \qquad x \in \mathbb{R}.$$

In this case we have

$$\partial G = \{(x, h_1(x)) \,:\, x \in \mathbb{R}\} \cup \{(x, h_2(x)) \,:\, x \in \mathbb{R}\},$$

and, for any $x \in \mathbb{R}$,

$$\nu(x, h_i(x)) = (1 + |h_i'(x)|^2)^{-\frac{1}{2}}((-1)^i h_i'(x), (-1)^{i+1}), \qquad i = 1, 2.$$

The corresponding graph Γ consists of just one edge $I_1 = \mathbb{R}$ and

$$\Pi_\epsilon(t) = \Pi(Z^\epsilon(t)) = (X^\epsilon(t), 1).$$

Moreover, the limiting process $\bar{Z}(t)$ is the solution of the stochastic equation

$$d\bar{Z}(t) = \frac{1}{2}\frac{\alpha'(\bar{Z}(t))}{\alpha(\bar{Z}(t))}\,dt + dB(t), \qquad \bar{Z}(0) = x.$$

Now, for any $\epsilon, \gamma > 0$, we consider the stochastic Skorokhod problem

$$\begin{cases} dZ^{\epsilon,\gamma}(t) = \sqrt{\hat{\sigma}_\epsilon}\, dB(t) + \hat{\sigma}_\epsilon\, \nu(Z^{\epsilon,\gamma}(t))\, d\phi^{\epsilon,\gamma}(t), \\ \\ Z^{\epsilon,\gamma}(k\gamma) = Z^{\epsilon}(k\gamma),, \qquad t \in [k\gamma, (k+1)\gamma). \end{cases}$$

where

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Clearly, for any $t \in [k\gamma, (k+1)\gamma)$ the variable

$$Z^{\epsilon,\gamma}(t) = (X^\epsilon(k\gamma)), Y^{\epsilon,\gamma}(t))$$

lives in the random interval

$$C(X^\epsilon(k\gamma)) = [h_1(X^\epsilon(k\gamma)), h_2(X^\epsilon(k\gamma))].$$

We have shown that there exists $\kappa_1 > 0$ such that, if we set $\gamma_\epsilon = \epsilon^2 \log \epsilon^{-\kappa_1}$, for any $T > 0$ it holds

$$\lim_{\epsilon\to 0} \sup_{z\in G} \sup_{t\in[0,T]} \mathbb{E}_z \left|Z^\epsilon(t) - Z^{\epsilon,\gamma_\epsilon}(t)\right|^2 = 0.$$

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- We show that for any $p \geq 1$ there exists $c_p > 0$ such that for any $\epsilon, \gamma > 0$, $k \in \mathbb{N}$ and $t, s \in [k\gamma, (k+1)\gamma)$

$$\sup_{z \in G} \mathbb{E}_z \left|\phi^{\epsilon,\gamma}(t) - \phi^{\epsilon,\gamma}(s)\right|^p \leq c_p \left(\gamma^p + \epsilon^p \gamma^{p/2} + \epsilon^{2p}\right).$$

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- We apply Ito's formula to a suitable function involving the exponential of an extension of distance function from ∂G.

About the proof of limit (6). Step 1

For every $0 < \delta' < \delta$ and $\epsilon > 0$ sufficiently small

$$\left|\mathbb{E}_z\ \left(\varphi(Z^\epsilon(t)) - (\varphi^\wedge)^\vee(Z^\epsilon(t))\right)\right| \leq c_T\ \left(\frac{\|\varphi\|_\infty}{\tau}\,\delta + \frac{1}{\delta}\,L^\epsilon_{\delta,\delta'}(t)\right),$$

for every $t \in\ [\tau, T]$, where

$$L^\epsilon_{\delta,\delta'}(t) := \sup_{z\in C(\delta)}\left|\mathbb{E}_z\ \left(\varphi(Z^\epsilon(t)) - (\varphi^\wedge)^\vee(Z^\epsilon(t))\,;\ \tau_1^{\epsilon,\delta,\delta'} > k^\epsilon_t\gamma_\epsilon\right)\right|,$$

and

$\tau_1^{\epsilon,\delta,\delta'}$ is the first time $Z_\epsilon(t)$ is at distance δ' of one of the interior vertices starting from distance at most δ.

Here $k^\epsilon_t \in\ \mathbb{N}$ such that $\frac{t}{2} < k^\epsilon_t\gamma_\epsilon < t$.

Step 2

For every $0 < \delta' < \delta$ it holds

$$\lim_{\epsilon \to 0} \sup_{t \in [\tau, T]} L^{\epsilon}_{\delta,\delta'}(t) = 0.$$

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For any $\epsilon > 0$ we have

$$\varphi(Z^\epsilon(t)) - (\varphi^\wedge)^\vee(Z^\epsilon(t)) = \left[\varphi(Z^\epsilon(t)) - \varphi(\hat{Z}^\epsilon(t))\right]$$

$$+ \left[\varphi(\hat{Z}^\epsilon(t)) - (\varphi^\wedge)^\vee(\hat{Z}^\epsilon(t))\right] + \left[(\varphi^\wedge)^\vee(\hat{Z}^\epsilon(t)) - (\varphi^\wedge)^\vee(Z^\epsilon(t))\right]$$

$$=: \sum_{i=1}^{3} I_i^\epsilon(t).$$

Due to the approximation result we have

$$\lim_{\epsilon\to 0}\sup_{z\in C(\delta)}\sup_{t\in[\tau,T]}\mathbb{E}_z\left(|I_i^\epsilon(t)|\,;\,\tau_1^{\epsilon,\delta,\delta'}>k_t^\epsilon\gamma_\epsilon\right)=0,\quad i=1,3.$$

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And because of averaging

$$\lim_{\epsilon\to 0}\sup_{t\in[\tau,T]}\mathbb{E}_z\left(I_2^\epsilon(t)\,;\,\tau_1^{\epsilon,\delta,\delta'}>k_t^\epsilon\gamma_\epsilon\right)=0.$$

Limit (7)

The limit

$$\lim_{\epsilon\to 0} \sup_{t\in[\tau,T]} \left|\mathbb{E}_z \varphi^\wedge(\Pi_\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(z)}\varphi^\wedge(\bar{Z}(t))\right| = 0,$$

would be a consequence of

$$\lim_{\epsilon\to 0} \mathbb{E}_z F(\Pi_\epsilon(\cdot)) = \bar{\mathbb{E}}_{\Pi(z)} F(\bar{Z}(\cdot)),$$

if for any $\varphi \in C(\bar{G})$, the function $\varphi^\wedge$ were a continuous function on $\bar{\Gamma}$.

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Unfortunately, in general $\varphi^\wedge$ is not continuous at the internal vertices of Γ, so that the proof of (7) requires a thorough analysis, which also involves a few estimates of the exit times of the process $Z_\epsilon(t)$ from the small neighborhoods of the points $(x,y) \in \partial G$ where $\nu_2(x,y) = 0$ described above.

From the SPDE on G to the SPDE on the graph Γ

Let us consider the SPDE on G

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t}(t,x,y) = \mathcal{L}_\epsilon u_\epsilon(t,x,y) + b(u_\epsilon(t,x,y)) + \dfrac{\partial w^Q}{\partial t}(t,x,y), \\ \\ \nabla u_\epsilon(t,x,y)\cdot \sigma_\epsilon \nu(x,y) = 0, \quad (x,y)\in \partial G, \qquad u_\epsilon(0,x,y) = u_0(x,y). \end{cases}$$

Our purpose here is to study the limiting behavior of its unique mild solution u_ϵ in the space $L^p(\boldsymbol{\Omega}; C([0,T];H))$, as $\epsilon \to 0$.

We are assuming here that

$$w^Q(t) = \sum_{k=1}^{\infty} Q e_k \beta_k(t), \qquad t \geq 0,$$

where $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis in H, $\{\beta_k(t)\}_{k\in\mathbb{N}}$ is a sequence of independent Brownian motions and

Q is a Hilbert-Schmidt linear operator in H.

Moreover, we set $B(x)(\xi) = b(x(\xi))$, for every $x \in H$ and $\xi \in G$.

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Recall that an $\mathcal{F}_t$-adapted process $u_\epsilon \in L^p(\mathbf{\Omega}; C([0,T]; H))$ is a mild solution for the problem above if

$$u_\epsilon(t) = S_\epsilon(t)u_0 + \int_0^t S_\epsilon(t-s)B(u_\epsilon(s))\,ds + \int_0^t S_\epsilon(t-s)dw^Q(s).$$

The SPDE on the graph Γ

Let us consider the problem

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t}(t,x,k) = \bar{L}\bar{u}(t,x,k) + b(\bar{u}(t,x,k)) + \dfrac{\partial \bar{w}^Q}{\partial t}(t,x,k), \\ \bar{u}(0,x,k) = {u_0}^{\wedge}(x,k), \end{cases} \tag{8}$$

where $u_0 \in C(\bar{G})$ and $\bar{L}$ is the limiting generator on Γ from Freidlin and Wentcell.

The SPDE on the graph Γ

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where $u_0 \in C(\bar{G})$ and $\bar{L}$ is the limiting generator on Γ from Freidlin and Wentcell.

Here $\bar{w}^Q$ is the cylindrical Wiener process defined by

$$\bar{w}^Q(t) = \sum_{j=1}^{\infty} (Qe_j)^\wedge \, \beta_j(t).$$

It is easy to check that

$$\mathbf{E}\langle \bar{w}^Q(t), f\rangle_{\bar{H}}\langle \bar{w}^Q(s), g\rangle_{\bar{H}} = \langle (QQ^\star f^\vee)^\wedge, g\rangle_{\bar{H}}(t \wedge s).$$

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Notice that if $Q \in \mathcal{L}_2(H)$, then

$$\sum_{j=1}^\infty \langle (QQ^\star f_j^\vee)^\wedge, f_j\rangle_{\bar{H}} = \sum_{j=1}^\infty \langle QQ^\star f_j^\vee, f_j^\vee\rangle_H \leq \|Q\|^2_{\mathcal{L}_2(H)} < \infty,$$

so that $\bar{w}^Q(t) \in L^2(\boldsymbol{\Omega}; \bar{H})$, for any $t \geq 0$, and defines a $\bar{H}$-valued Wiener process, with covariance operator

$$(QQ^\star)^\wedge f := (QQ^\star f^\vee)^\wedge, \quad f \in \bar{H}.$$

As we have seen, the operator $\bar{L}$ is the generator of the Markov transition semigroup $\bar{S}(t)$ associated with the limiting process $\bar{Z}(t)$ defined on the graph Γ.

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Thus, we can say that

An adapted process $\bar{u} \in L^p(\boldsymbol{\Omega}; C([0,T]; \bar{H}))$ is a *mild solution* to equation (8) if

$$\bar{u}(t) = \bar{S}(t)u_0^\wedge + \int_0^t \bar{S}(t-s)B(\bar{u}(s))\,ds + \int_0^t \bar{S}(t-s)d\bar{w}^Q(s).$$

As we are assuming $Q \in \mathcal{L}_2(H)$, then $\bar{w}^Q(t) \in L^2(\mathbf{\Omega}, \bar{H})$, for any $t \geq 0$. Moreover, $\bar{S}(t)$ is a contraction on $\bar{H}$, so that the process $w_{\bar{L}}(t)$ defined by

$$w_{\bar{L}}(t) := \int_0^t \bar{S}(t-s)d\bar{w}^Q(s), \quad t \geq 0,$$

takes values in $L^p(\mathbf{\Omega}; C([0, T]; \bar{H}))$, for any $T > 0$ and $p \geq 1$.

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Therefore, as the mapping $B : \bar{H} \to \bar{H}$ is Lipschitz-continuous, we have that for any $T > 0$ and $p \geq 1$

there exists a unique mild solution $\bar{u} \in L^p(\boldsymbol{\Omega}; C([0,T]; \bar{H}))$ to equation (8).

The limit theorem

Assume that the domain G satisfies assumptions I-IV. Moreover, assume that the nonlinearity $b : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous and $Q \in \mathcal{L}_2(H)$.

Then, for any $u_0 \in C(\bar{G})$, $p \geq 1$ and $0 < \tau < T$ we have

$$\lim_{\epsilon \to 0} \mathbf{E} \sup_{t \in [\tau, T]} |u_\epsilon(t) - \bar{u}(t)^\vee|_H^p$$

$$= \lim_{\epsilon \to 0} \mathbf{E} \sup_{t \in [\tau, T]} |u_\epsilon(t)^\wedge - \bar{u}(t)|_{\bar{H}}^p = 0,$$

where u_ϵ and $\bar{u}$ are the unique mild solutions of the SPDE on G and of the SPDE on Γ, respectively.

A few words about the proof

We have

$$u_\epsilon(t) = S_\epsilon(t)u_0 + \int_0^t S_\epsilon(t-s)B(u_\epsilon(s))\,ds + w_\epsilon(t),$$

where

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where

$$w_\epsilon(t) = \int_0^t S_\epsilon(t-s)dw^Q(s).$$

This implies that for any $p \geq 1$ and $T > 0$

$$|u_\epsilon(t) - \bar{u}(t)^\vee|_H^p \leq c_p\,|S_\epsilon(t)u_0 - \bar{S}(t)^\vee u_0|_H^p$$

$$+c_p\,T^{p-1}\int_0^t |S_\epsilon(t-s)B(u_\epsilon(s)) - (\bar{S}(t-s)B(\bar{u}(s)))^\vee|_H^p\,ds$$

$$+c_p\,\left|w_\epsilon(t) - w_{\bar{L}}(t)^\vee\right|_H^p.$$

This implies

$$|u_\epsilon(t) - \bar{u}(t)^\vee|_H^p \leq R_p^\epsilon(t) + c_p \int_0^t |u_\epsilon(s) - \bar{u}(s)^\vee|_H^p \, ds,$$

where

$$R_p^\epsilon(t) := c_p \, |S_\epsilon(t) u_0 - \bar{S}(t)^\vee u_0|_H^p + c_p \, \left| w_\epsilon(t) - w_{\bar{L}}(t)^\vee \right|_H^p$$

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$$+ c_p \int_0^t |S_\epsilon(t-s) B(\bar{u}(s))^\vee - (\bar{S}(t-s) B(\bar{u}(s)))^\vee|_H^p \, ds.$$

Therefore, our result follows once we prove that

$$\lim_{\epsilon \to 0} \mathbf{E} \sup_{t \in [0,T]} |w_\epsilon(t) - w_{\bar{L}}(t)^\vee|_H^p = 0.$$

Thank You

We have

$$Q_\epsilon = I_{\epsilon,1} Q,$$

where for any $\epsilon_1, \epsilon_1 > 0$ and $Q \in \mathcal{L}(H_{\epsilon_1})$, we have defined

$$I_{\epsilon_2,\epsilon_1} Q = J_{\epsilon_2,\epsilon_1} \circ Q \circ J_{\epsilon_1,\epsilon_2} \in \mathcal{L}(H_{\epsilon_2}),$$

with

$$J_{\epsilon_2,\epsilon_1} f(x,y) = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\, f(x, \epsilon_1 \epsilon_2^{-1} y), \qquad (x,y) \in G_{\epsilon_2},$$

for every $f \in H_{\epsilon_1}$.

Gluing conditions

For any vertex $O_i = (x_i, k_1) = \cdots = (x_i, k_{N_i})$ there exist finite

$$\lim_{(x,k_j)\to O_i} \bar{L}f(x, k_j),$$

the following one-sided limits exist

$$\lim_{x\to x_i} l_k(x)\frac{df}{dx}(x, k_j),$$

along any edge I_{k_j} ending at the vertex $O_i = (x_i, k_j)$ and the following gluing condition is satisfied

$$\sum_{k=1}^{N_i}\left(\pm \lim_{x\to x_i} l_k(x)\frac{df}{dx}(x, k)\right) = 0,$$

where the sign $+$ is taken for right limits and the sign $-$ for left limits. In the case of an exterior vertex O_i, the gluing condition (43) reduces to

$$\lim_{x\to x_i} l_k(x)\frac{df}{dx}(x, k) = 0,$$