Renormalisation in regularity structures

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(joint work with Yvain Bruned and Martin Hairer)

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Renormalisation

From Wikipedia, the free encyclopedia

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[L.Z. : whatever this means...]

Models, parameters, predictions

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We should change our model ! Namely find $R_\varepsilon : \mathcal{M} \to \mathcal{M}$ such that

$$\hat{\Phi} : \mathcal{M} \to \mathcal{P}, \qquad \hat{\Phi}(m) = \lim_{\varepsilon \to 0} \Phi_\varepsilon \circ R_\varepsilon(m).$$

The stochastic heat equation

Let $v : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ solve the heat equation with external forcing

$$\partial_t v = \Delta v + \xi, \qquad x \in \mathbb{R}^d.$$

where $\xi = \xi(t,x)$ is a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^d$, i.e. a centered Gaussian field such that

$$\mathbb{E}(\xi(x,t)\xi(y,s)) = \delta(x-y)\,\delta(t-s), \qquad t,s \geq 0,\ x,y \in \mathbb{R}^d.$$

A concrete realisation: for all $\psi \in L^2(\mathbb{R}^d)$ and $t \geq 0$

$$\int_{[0,t]\times\mathbb{R}^d} \psi(x)\,\xi(s,x)\,ds\,dx := \sum_k B_k(t)\,\langle e_k, \psi\rangle,$$

where $(B_k)_k$ is an IID sequence of Brownian motions and $(e_k)_k$ is a complete orthonormal system in $L^2(\mathbb{R}^d)$.

The stochastic heat equation

The stochastic heat equation

$$\partial_t v = \Delta v + \xi, \qquad x \in \mathbb{R}^d$$

has a unique solution given by

$$v(t,x) = \int G_t(x-y)\, v(0,y)\, dy + \int G_{t-s}(x-y)\, \xi(ds,dy)$$

where G is the heat kernel.

The path properties of this ”process” depend heavily on the dimension, since for $v(0,\cdot) = 0$

$$\mathbb{E}((v(t,x))^2) = \int (G_{t-s}(y))^2 ds\, dy = \int_0^t \frac{C_d}{s^{\frac{d}{2}}}\, ds \begin{cases} < +\infty, & d = 1 \\ = +\infty, & d \geq 2 \end{cases}$$

Random distributions

Therefore v is a well-defined process only for $d = 1$. For $d \geq 2$ it makes sense as a random field: for all $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\mathbb{E}(\langle \varphi, v(t,\cdot)\rangle^2) = \int \varphi(x)\, G_{2(t-s)}(x-x')\, \varphi(x')\, ds\, dx\, dx'$$

which is finite for all $d \geq 1$. This random field is a.s. $C(\mathbb{R}_+, H^{1-\frac{d}{2}-\kappa})$ for all $\kappa > 0$.

In particular if we want to study equations like

$$\partial_t u = \Delta u + F(u) + \xi, \qquad x \in \mathbb{R}^d$$

we write the equation in the mild form

$$u(t,x) = \int G_t(x-y)\, u(0,y)\, dy + \int G_{t-s}(x-y)\, \xi(ds,dy) + \int G_{t-s}(x-y)\, F(u)(s,y)\, ds\, dy$$

and u has the same regularity as v. This is a problem if F is non-linear.

Singular stochastic PDEs

$$\partial_t u = \Delta u + F(u, \nabla u, \xi), \qquad x \in \mathbb{R}^d$$

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Even for polynomial non-linearities, we do not know how to properly define products of (random) distributions.

This is where infinities arise.

Regularisation

Let $\xi_\varepsilon = \rho_\varepsilon * \xi$ a regularisation of ξ and let u_ε solve

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + F(u_\varepsilon, \nabla u_\varepsilon, \xi_\varepsilon), \qquad x \in \mathbb{R}^d.$$

What happens as $\varepsilon \to 0$?

If we fix a Banach space of generalised functions $\mathcal{H}^{-\alpha}$ on $\mathbb{R}^{d+1}$ such that $\xi \in \mathcal{H}^{-\alpha}$ a.s. for some fixed $\alpha > 0$, then the map $\xi_\varepsilon \mapsto u_\varepsilon$ is not continuous.

We need a topology such that

- the map $\xi_\varepsilon \mapsto u_\varepsilon$ is continuous
- $\xi_\varepsilon \to \xi$ as $\varepsilon \to 0$.

The theory of regularity structures (RS) considers these two problems separately.

Factorisation

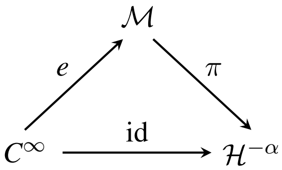
More precisely, the RS theory gives (for a class of equations)

- a metric space $(\mathcal{M}, d)$
- a non-linear canonical embedding

$$C^\infty(\mathbb{R}^{d+1}) \ni \zeta \mapsto e(\zeta) \in \mathcal{M}$$

- a canonical surjective projection $\pi : \mathcal{M} \to \mathcal{H}^{-\alpha}$ such that

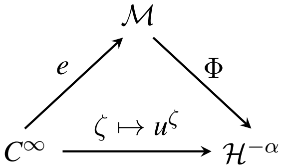
$$\pi \circ e(\zeta) = \zeta, \qquad \forall \zeta \in C^\infty(\mathbb{R}^{d+1}).$$



Factorisation

Moreover we have

- a continuous map $\Phi : \mathcal{M} \to \mathcal{H}^{-\alpha}$ such that if for $\zeta \in C^\infty$, u^ζ is defined by
$$\partial_t u^\zeta = \Delta u^\zeta + F(u^\zeta, \nabla u^\zeta, \zeta)$$
then
$$\Phi \circ e(\zeta) = u^\zeta$$



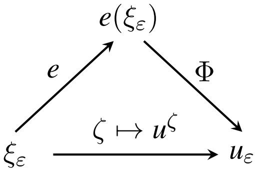
In this diagram, all elements are canonical. However there is no canonical extension $e : \mathcal{H}^{-\alpha} \to \mathcal{M}$ since e is not continuous in the topology of $\mathcal{H}^{-\alpha}$.

Factorisation

Now we can write our regularised SPDE as follows

$$u_\varepsilon = \Phi \circ e(\xi_\varepsilon)$$

where ξ is white noise and $\xi_\varepsilon = \rho_\varepsilon * \xi$.



The convergence problem factorises in two separate problems:

- (Analytic step) Construction of $(\mathcal{M}, d)$ and continuity of Φ.
- (Probabilistic step) Convergence of $e(\xi_\varepsilon)$ as $\varepsilon \to 0$ to an $\mathcal{M}$-valued random variable that we call $e(\xi)$.

Polynomials

For $\zeta \in C^\infty$, $e(\zeta) \in \mathcal{M}$ is given by

$$e(\zeta) = (P_1(\zeta), \dots, P_K(\zeta))$$

where the P_i's are polynomial functionals of ζ. This family (in particular K) depends on the equation.

Examples:

$$\zeta, \qquad \zeta(G * \zeta), \qquad (\partial_x G * \zeta)^2, \qquad \zeta\, G * (\zeta\, G * \zeta).$$

Convergence in $\mathcal{M}$ means (roughly) convergence of $P_i(\zeta)$, $i = 1, \dots, K$, as generalised functions.

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Question: does $P_i(\rho_\varepsilon * \xi)$ converge as $\varepsilon \to 0$?

One polynomial

Let $\varphi \in C_c^\infty$. Then we define $z := (t,x)$ and for the polynomial $\zeta(G * \zeta)$

$$T_\varepsilon := \int \varphi(z)\, \xi_\varepsilon(z)\, (G * \xi_\varepsilon)(z)\, dz.$$

Now

$$\mathbb{E}[T_\varepsilon] = \int \varphi(z)\, \mathbb{E}[\xi_\varepsilon(G * \xi_\varepsilon)](z)\, dz = \int \varphi(z)\, \rho_\varepsilon * G * \rho_\varepsilon(0)\, dz$$

and

$$\lim_{\varepsilon \to 0} \mathrm{Var}[T_\varepsilon] = \int \varphi^2(z)\, G^2(z-x)\, dz\, dx < +\infty.$$

However $\rho_\varepsilon * G * \rho_\varepsilon(0) \to +\infty$ as $\varepsilon \to 0$: a first example of the famous infinities.

Lack of convergence

In fact as soon as P_i is non-linear, $P_i(\xi_\varepsilon)$ tends not to converge as $\varepsilon \to 0$, even as a Schwartz distribution.

Therefore, our

- (Probabilistic step) If ξ is white noise and $\xi_\varepsilon = \rho_\varepsilon * \xi$ then $e(\xi_\varepsilon)$ converges to an $\mathcal{M}$-valued random variable that we call $e(\xi)$

seems fo fail.

In particular there is no canonical $e(\xi)$.

However if $e(\xi_\varepsilon)$ does not converge, how can $u_\varepsilon = \Phi \circ e(\xi_\varepsilon)$?

The fibre

Important remarks:

- $e(\zeta) \in \mathcal{M}$ and $\pi \circ e(\zeta) = \zeta$, but there can be other elements $Z \in \pi^{-1}(\zeta)$, namely such that $\pi(Z) = \zeta$.
- $Z \in \pi^{-1}(\zeta)$ contains other possible (non-canonical) definitions of $\zeta(G * \zeta)$, $(\partial_x G * \zeta)^2$ etc.
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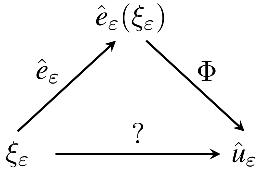
Apparently we have to modify our $e(\xi_\varepsilon) = (P_i(\xi_\varepsilon))_{i=1,\dots,K}$.

That means choosing another $\hat{e}_\varepsilon(\xi_\varepsilon) \in \pi^{-1}(\xi_\varepsilon)$ (see slide 9)

If $e(\xi_\varepsilon)$ becomes $\hat{e}_\varepsilon(\xi_\varepsilon)$, then u_ε becomes $\hat{u}_\varepsilon := \Phi \circ \hat{e}_\varepsilon(\xi_\varepsilon)$.



Renormalisation

In our example it is reasonable to modify $\xi_\varepsilon(G * \xi_\varepsilon)$ into

$$\xi_\varepsilon(G * \xi_\varepsilon) - \mathbb{E}[\xi_\varepsilon(G * \xi_\varepsilon)].$$

We have renormalised this product.

The modification should change as little as possible the solution.

The non-linear character of $\mathcal{M}$ imposes constraints on the possible modifications (and viceversa).

We want at least that

$$\pi \circ \hat{e}_\varepsilon(\xi_\varepsilon) = \xi_\varepsilon, \qquad \lim_{\varepsilon \to 0} \hat{e}_\varepsilon(\xi_\varepsilon) = \hat{e}(\xi) \qquad \text{in } \mathcal{M}.$$

Then

$$\lim_{\varepsilon \to 0} \hat{u}_\varepsilon = \lim_{\varepsilon \to 0} \Phi \circ \hat{e}_\varepsilon(\xi_\varepsilon) =: \hat{u} \qquad \text{in } \mathcal{H}^{-\alpha}.$$

Convergence

Questions:

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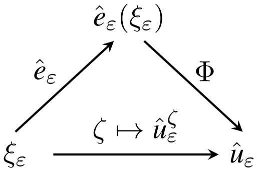
One can define for instance

$$\xi_\varepsilon(G * \xi_\varepsilon) \mapsto \xi_\varepsilon(G * \xi_\varepsilon) - \mathbb{E}[\xi_\varepsilon(G * \xi_\varepsilon)] + c$$

for any constant $c \in \mathbb{R}$ and this still defines a good $\hat{e}$.

Diagrams

Moreover it is possible to choose $\hat{e}_\varepsilon(\xi_\varepsilon)$

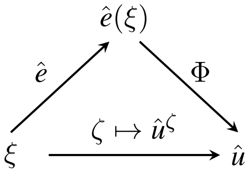


where $C^\infty \ni \zeta \mapsto \hat{u}^\zeta_\varepsilon \in C^\infty$ is given by

$$\partial_t \hat{u}^\zeta_\varepsilon = \Delta \hat{u}^\zeta_\varepsilon + \hat{F}_\varepsilon(\hat{u}^\zeta_\varepsilon, \nabla \hat{u}^\zeta_\varepsilon, \zeta).$$

Diagrams

In the limit $\varepsilon \to 0$ we obtain $\hat{e}(\xi)$



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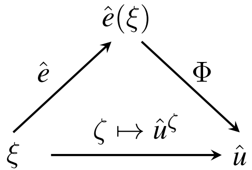

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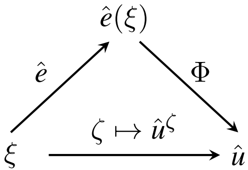

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This is like a homogeneisation result.

However $\hat{F}$ is very different from F.

Renormalisation group

For a class of equations we have

Theorem

There exists a finite-dimensional Lie group $\mathfrak{R}$ acting on $\mathcal{M}$ and deterministic $R_\varepsilon \in \mathfrak{R}$ such that $\hat{e}_\varepsilon(\xi_\varepsilon) = R_\varepsilon\, e(\xi_\varepsilon)$, namely

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\quad R_\varepsilon \quad} & \mathcal{M} \\ e \uparrow & & \uparrow \hat{e}_\varepsilon \\ \xi_\varepsilon & \xrightarrow{\quad \mathrm{id} \quad} & \xi_\varepsilon \end{array}$$

Moreover if $\hat{e}^i_\varepsilon(\xi_\varepsilon)$ converges for $i = 1, 2$, then $(R^1_\varepsilon)^{-1} R^2_\varepsilon \to R \in \mathfrak{R}$. Therefore $\mathfrak{R}$ parametrises the possible renormalised solutions $\hat{u}$.

Remember that $\hat{u}_\varepsilon$ solves a modified equation with non-linearity $\hat{F}_\varepsilon$. Then $\mathfrak{R}$ has a dual action $F \mapsto \hat{F}_\varepsilon$.

Examples of renormalised equations

$$\text{(KPZ)} \qquad \partial_t \hat{u}_\varepsilon = \Delta \hat{u}_\varepsilon + (\partial_x \hat{u}_\varepsilon)^2 - C_\varepsilon + \xi_\varepsilon, \quad x \in \mathbb{R},$$

$$\text{(gKPZ)} \qquad \begin{aligned} \partial_t \hat{u}_\varepsilon = \Delta \hat{u}_\varepsilon + f(\hat{u}_\varepsilon)\left((\partial_x \hat{u}_\varepsilon)^2 - C_\varepsilon\right) + h(\hat{u}_\varepsilon) \\ + g(\hat{u}_\varepsilon)\left(\xi_\varepsilon - C_\varepsilon g'(\hat{u}_\varepsilon)\right), \quad x \in \mathbb{R}, \end{aligned}$$

$$\text{(PAM)} \qquad \partial_t \hat{u}_\varepsilon = \Delta \hat{u}_\varepsilon + \hat{u}_\varepsilon \, \xi_\varepsilon - C_\varepsilon, \quad x \in \mathbb{R}^2,$$

$$(\Phi^4_3) \qquad \partial_t \hat{u}_\varepsilon = \Delta \hat{u}_\varepsilon - \hat{u}^3_\varepsilon + (C^1_\varepsilon + C^2_\varepsilon)\, \hat{u}_\varepsilon + \xi_\varepsilon, \quad x \in \mathbb{R}^3.$$

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Now, most of this can be found in Martin's paper

- M. Hairer (2014), *A theory of regularity structures*. Invent. Math.

However in this paper the renormalisation group has to be guessed.

The beginning of my talk

Remember that $e(\zeta) = (P_1(\zeta), \dots, P_K(\zeta))$.

We have now K random variables $P_1(\xi_\varepsilon), \dots, P_K(\xi_\varepsilon)$, polynomial functions of ξ_ε.

More precisely, for a fixed $\varphi \in C_c^\infty$ we consider the random variables

$$X_i := \int_{\mathbb{R}^{d+1}} \varphi(z)\, P_i(\xi_\varepsilon(z))\, dz, \qquad i = 1, \dots, K.$$

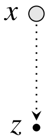
To each such random variable we associate a rooted tree T_i.

Every integration variable in X_i is a vertex in T_i.

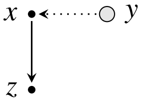
Every integral kernel in X_i is an edge in T_i.

Examples

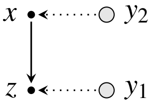
$$\int \varphi(z)\, \xi_\varepsilon(z)\, dz = \int \varphi(z)\, \rho_\varepsilon(z-x)\, \xi(dx)\, dz \quad \longrightarrow$$



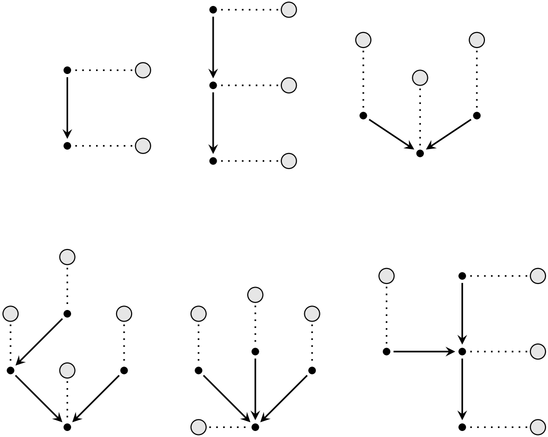
$$\int \varphi(z)\, G * \xi_\varepsilon(z)\, dz \quad \longrightarrow$$



$$\int \varphi(z)\, \xi_\varepsilon(z)\, G * \xi_\varepsilon(z)\, dz \quad \longrightarrow$$



Examples

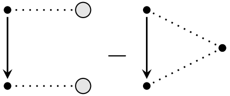


Feynman diagrams

Do you remember? We noticed that $\xi_\varepsilon\, G * \xi_\varepsilon$ can be renormalised by subtracting its expectation:

$$\xi_\varepsilon\, G * \xi_\varepsilon - \mathbb{E}[\xi_\varepsilon\, G * \xi_\varepsilon] = \xi_\varepsilon\, G * \xi_\varepsilon - \rho_\varepsilon * G * \rho_\varepsilon(0).$$

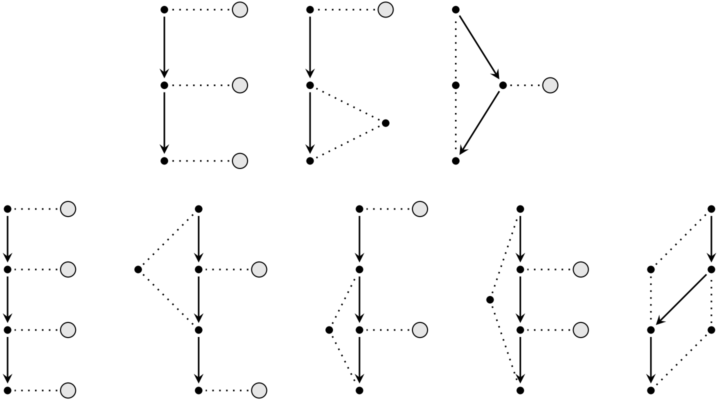
In terms of graphs (Feynman diagrams), this can be written as



Note that graphically the second graph is obtained from the first after a contraction of two leaves.

Feynman diagrams

Other contractions:



Negative trees

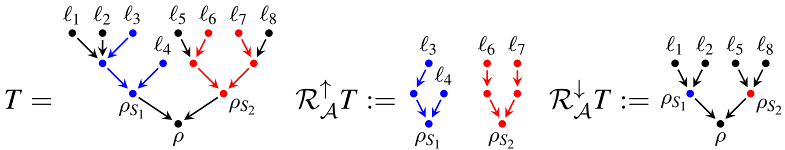
Some of these contractions create diverging quantities, some do not.

When a diverging pattern appears, we call the subtree containing the contracted leaves negative.

Given a rooted tree T, we call

$$\mathfrak{A}(T) := \{(S_1, \ldots, S_n) : S_i \subset T \text{ negative subtree}, \ S_i \cap S_j = \emptyset, \ i \neq j\}$$

and for $\mathcal{A} \in \mathfrak{A}(T)$ we define a forest $\mathcal{R}^{\uparrow}_{\mathcal{A}} T$ and a tree $\mathcal{R}^{\downarrow}_{\mathcal{A}} T$



Renormalisation group

Then we have a general systematic description of $\mathfrak{R}$.

Theorem

The renormalisation group $\mathfrak{R}$ *is given by*

$$M_\ell T := \sum_{\mathcal{A}\in\mathfrak{A}(T)} \ell(\mathcal{R}^{\uparrow}_{\mathcal{A}}T)\,\mathcal{R}^{\downarrow}_{\mathcal{A}}T$$

where ℓ *is a multiplicative functional on forests*

$$\ell(S_1,\ldots,S_n) = \prod_{i=1}^{n} \ell(S_i).$$

We have an explicit description of the product and the inverse in $\mathfrak{R}$:

$$M_\ell M_{\ell'} = M_{\ell\circ\ell'}, \qquad (M_\ell)^{-1} = M_{\ell''}$$

based on Hopf Algebras of Trees. Note that $\mathfrak{R}$ depends on the equation.

Convergence

We also need a general systematic result on convergence of renormalised models.

The following result is still in progress

Theorem

- *There exists a family* $(\ell(\varepsilon))$ *such that*

$$\hat{e}_\varepsilon(\xi_\varepsilon) := M_{\ell(\varepsilon)} e(\xi_\varepsilon)$$

  converges in $\mathcal{M}$ *as* $\varepsilon \to 0$.
- *All possible limits of* $\hat{e}_\varepsilon(\xi_\varepsilon)$ *(and therefore of* $\hat{u}_\varepsilon$*) are parametrised by* $\mathfrak{R}$.

The end

Thanks !

Sorry !