10/29. Thomas Duyckaerts

Joint with Tristan Roy

(NLW) $\begin{cases} \partial_t^2 u - \Delta u = |u|^{p-1} u \\ \vec{u}|_{t=0} = (u_0, u_1) \end{cases}$ $\qquad x \in \mathbb{R}^3$

$p > 5 \qquad \vec{f} = (f, \partial_t f)$

Locally well-posed in $\dot{\mathcal{H}}^{s_c} = \dot{H}^{s_c}(\mathbb{R}^3) \times \dot{H}^{s_c-1}(\mathbb{R}^3)$

$s_c = \frac{3}{2} - \frac{2}{p-1}$ conserved energy

$$E(\vec{u}) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla_{t,x} u|^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}$$

$$|\nabla_{t,x} u|^2 = |\nabla_x u|^2 + |\nabla_t u|^2$$

$p > 5 \iff s_c > 1 \qquad p = 5 \iff s_c = 1$

Finite speed of propagation.

$u(t,x)$ is determined by $\vec{u}(0,y)$ $\quad |y-x| < |t|$

- Profile decomposition (Bahouri - Gérard
- Rigidity classification of non dispersive objects.
  - Merle 1995
- exterior energy estimate (T.D. Kenig. Merle)
- no soliton

II Typical examples of solutions

A Type I blow-up solutions

Def. u is a type I blow-up solution if $T_+ < \infty$ and

$$\lim_{t \to T_+} \|\vec{u}(t)\|_{\dot{\mathcal{H}}^{s_c}} = +\infty$$

ODE

$$y'' = |y|^{p-1} y \qquad y(t) = \frac{c_p}{(T-t)^{\frac{2}{p-1}}}$$

Merle and Zaag   Donninger   and   Schörkhuber

B   Linear equation.

$$(LW)\begin{cases}\partial_t^2 u - \Delta u = 0\\ \vec{u}|_{t=0} = (u_0, u_1) \in \dot{\mathcal{H}}^{s_c}\end{cases}$$

Strichartz   (Ginibre and Velo '95).

$(u_0, u_1) \in \dot{\mathcal{H}}^{s_c} \Rightarrow v \in L^{2(p-1)}(\mathbb{R}\times\mathbb{R}^3)$.

Prop. If u is a solution of (NLW)

(i) $u \in L^{2(p-1)}([0,+\infty)_t \times \mathbb{R}^3_x)$

(ii) $\exists$ v solution of (LW) such that

$$\lim_{t\to+\infty} \|\vec{u}(t) - \vec{v}(t)\|_{\dot{\mathcal{H}}^{s_c}} = 0$$

"scattering".

small data scattering.

C.   Elliptic equation

(Ell) $\Delta Q = |Q|^{p-1} Q \qquad Q \in \dot{H}^{s_c}(\mathbb{R}^3)$

$p=5 \qquad W = \dfrac{1}{\left(1+\frac{|x|^2}{3}\right)^{1/2}} \sim \dfrac{1}{|x|}$ "only" radial solution except 0.

$p>5$

no solution in $\dot{H}^{s_c}(\mathbb{R}^3)$ Joseph / Lundgren 70's.

Prop There exist a solution of (Ell) for $n \neq 0$ such that

- $Q(n) \sim \dfrac{1}{n} \qquad n\to\infty$
- $Q \notin \dot{H}^{s_c}(\mathbb{R}^3)$

III.   Main Result

$p=5$ (Energy critical case)

"soliton resolution"   any non-type I blow-up solution decouples as $v(t)$ + finite sum of rescaled W

v: solution of (LW)

Conjecture for $p > 5$

Any solution of (NLW) is scattering or Type I blow-up.

Weaker version

(*) If $\limsup_{t \to +\infty} \|\vec{u}(t)\|_{\dot{H}^{s_c}} < \infty$, then u scatters.

(*) Kenig - Merle (2011) defocusing radial

Killip - Visan nonradial

TD KM (2014) focusing radial

higher dimension Bulut (2012)

Dodson Lawrie

- Navier - Stokes: Several works by Sverak, Kenig, Koch and coauthors

Thm (TD, Tristan Roy) If u is a solution of (NLW)

(i) $\lim_{t \to T_+} \|\vec{u}(t)\|_{\dot{H}^{s_c}} = +\infty$ or

(ii) $T_+ = +\infty$ and u scatters.

Krieger - Schlag constructed a global solution, non-scattering

$p = 7$ $s_c = \frac{7}{6}$ $\in \dot{H}^s,\ s > \frac{7}{6};\ \notin \dot{H}^{7/6}$

- Collot $D \geq 11$ $p \gg 1$

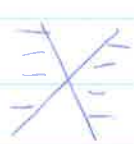
$T < +\infty$ $\dot{H}^s$ $s < s_c$ bounded, $\dot{H}^{s_c}$ blows up.

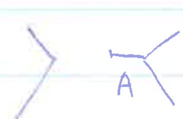
IV. Idea of proof.

A. Exterior energy estimates

Prop 1 Any solution v of (LW) satisfies $\forall t \geq 0$ or $t \leq 0$

$$\int_{|x| > |t|} |\nabla_{t,x} v(t,x)|^2 \, dx \geq \frac{1}{2} \int |\nabla v_0|^2 + v_1^2$$





Prop 2. If v is radial and $A > 0$, then

$$\int_{A+|t|}^{\infty} |\partial_r (rv)|^2 + |\partial_t (rv)|^2 \, dr \geq \frac{1}{2} \int_A^{+\infty} |\partial_{r,t} (rv)|^2 \Big|_{t=0}$$

Prop 2 $\Rightarrow$ Prop 1. $A \longrightarrow 0$

$$\int_B^{+\infty} (\partial_r (rv))^2 = \int_B^{+\infty} r^2 (\partial_r v)^2 - B v^2(B)$$

Prop 1. $(\partial_t^2 - \partial_r^2)(rv) = 0$

$$rv = f(t+r) - f(t-r)$$

$$(\partial_r (rv))^2 + (\partial_t (rv))^2 = r |f(t+r)|^2 + r |f(t-r)|^2$$

= modified energy" $m = \frac{p-1}{2}$.

$$E_m(\vec{v}(t)) = \int_0^{\infty} |\partial_r (rv)|^m + |\partial_t (rv)|^m \, dr$$

Prop

- $E_m(\vec{v}(t)) \approx E_m(\vec{v}(0))$
- $(v_0, v_1) \in \dot{\mathcal{H}}^{s_c} \Rightarrow E_m$ well-defined.
- exterior energy estimates.
- small data theory.

Assume $T_+ < \infty$.

$\|\vec{u}(t_n)\|_{\dot{\mathcal{H}}^{s_c}}$ bounded $t_n \to T_+$.

 $\vec{u}(t_n)$ decomposes as sum of profiles.