10/29 Nicolas Burq

w/ P. Gerard

M compact, g, $a \geq 0$

Δ $\quad \left(\frac{\partial^2}{\partial t^2} - \Delta\right) u + a(x) \partial_t u = 0 \qquad (*)$

$(u_0, u_1) \in H^1 \times L^2$

$E(u) = \frac{1}{2} \int_M |\nabla u|^2 + |\partial_t u|^2 \, dx$

IBP $\quad \frac{d}{dt} E = - \int_M a(x) |\partial_t u|^2$

i) $a \in L^\infty$,

$\begin{cases} -\Delta e = \lambda^2 e & \int_M a(x) |e|^2(x) = 0 \\ \Rightarrow e = 0 \end{cases}$

$\Rightarrow \forall (u_0, u_1) \quad E(t) \to 0$ as $t \to \infty$

$u = e^{it\lambda} e$ solution of $(*)$

ii) $E(t) \leq f(t) E(0)$, $\lim_{t\to\infty} f(t) = 0$

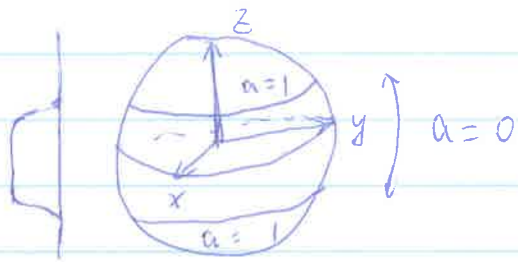
$\exists T$; $f(T) \leq \frac{1}{2}$

$\Rightarrow E(nT) \leq \left(\frac{1}{2}\right)^n$

$\Rightarrow f(t) \leq c\, e^{-ct}$

iii) An example

$M = S^2$



$-\Delta_g (x+iy)^n = n(n+1) e_n$

$\quad \| \; e_n$

$|e_n| \leq c\, e^{-cnz^2}$

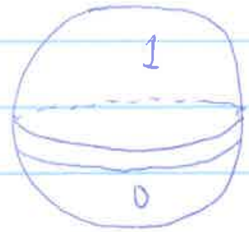
$\Delta(\chi(z) e_n) - (n(n+1)\chi(z) e_n = [\Delta, \chi] e_n = O(e^{-cn})$

Thm (Taylor, Rauch T. (Bardos, Lebeau, Rauch))

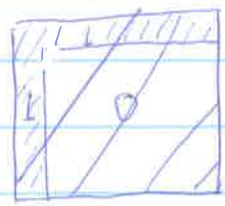
i) If $\exists T$, any geodesic $\gamma(s)$, $s \in [0, T]$ enters a

sGCC region where $a \geq c > 0$

⇒ Stabilization holds ($\exists C, c$. $E(t) \le C e^{-ct} E(0)$)



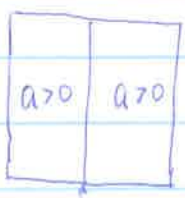
OK



WGCC.

ii) If $\forall T$ $\exists$ geodesic $\gamma(s), s \in [0,T]$ disjoint from supp(a), no stabilization.

Q.



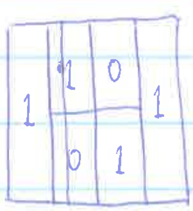
No SGCC

Yes WGCC

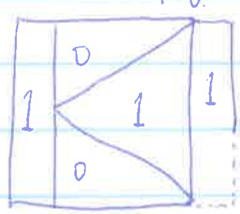
$a \in C^0$

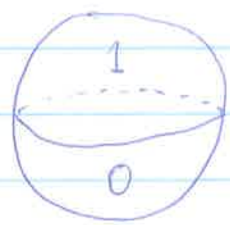
$a = 0$

Yes



No.





Thm (Lebeau)

Yes

Prop (N. B, P. Gerard)

$a \in C^0$. Then stabilization iff

GCC $\forall (x_0, \xi_0)$ $\gamma_{x_0,\xi_0}(s)$

$$\int_0^T a(\gamma_{x_0,\xi_0}(s)) \geq c > 0$$

Q. An anolog for $a \in L^\infty$

Thm (P. G., N. B.)

$\mathbb{T}^2$. $a = \sum_{j=1}^{N} a_j \mathbb{1}_{R_j}$ R_j rectangles. Then

Stab iff $\forall (x_0, \xi_0)$

$\gamma_{x_0,\xi_0}(s)$ either enters R_{j_0} for some j_0

or follows sides of R_{j_1}, R_{j_2} on right and on left.

II Proof of Prop.

1) Resolvent estimate

$P v = (-\Delta - \tau^2 + i a(x) \tau) v = g$

Stab $\Leftrightarrow$ P is invertible and

$\overset{\Leftarrow}{\text{easy}}$ $P v = f \Rightarrow \|v\|_{L^2} \leq \frac{C}{1+|\tau|} \|f\|_{L^2}$ $\tau \neq 0$.

1') Observation.

Res. est. $\Leftrightarrow$ $(-\Delta - \tau^2) u = f$

$\Rightarrow \|u\|_{L^2} \leq \frac{C}{1+|\tau|} \|f\|_{L^2} + D \|a^{1/2} u\|_{L^2}$

2)

Low frequency $|\tau| \leq C$ (OK)

3) High frequency $\tau \to \infty$ $\tau = h^{-1}$ $h \to 0$

$P_h = (-h^2 \Delta - 1) u = g$

$\|u\|_{L^2} \leq \frac{C}{h} \|g\|_{L^2} + D \|a^{1/2} u\|_{L^2}$

$P_h = b(h D_x)$

$b(\xi) = |\xi|^2 - 1$

$b \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$

$b(x, h D_x) u = \frac{1}{(2\pi)^d} \int e^{i(x-y)\cdot\xi} b(x, h\xi) u(y)\, dy\, d\xi$

$\overset{\shortparallel}{Op_h(b)}$ $u = \frac{1}{(2\pi)^d} \int e^{i(x-y)\cdot\xi} u(y)\, dy\, d\xi$

i)

$\|Op_h(b)\|_{\mathcal{L}(L^2)} \leq C$ w.r.t. h

ii) $Op_h(b_1)\,Op_h(b_2) = Op_h(b_1 \times b_2) + ih\,Op_h(\partial_\xi b_1 \cdot \partial_x b_2)$
$+ O(h^2)_{\mathcal{L}(L^2)}$

iii) $b \geq 0$

$Re(Op(b)) \geq -ch$ $\qquad T \geq \tilde{T}$

$|Im(Op(b))| \leq ch$ $\qquad (Tu,u)_{L^2} \geq (\tilde{T}u,u)_{L^2}$

$(-h^2\Delta - 1)\,u = f$

$\Rightarrow \|u\| \leq \frac{c}{h}\|f\| + D\|a^{1/2}u\|$. If not true,

$h_n \to 0$ $\quad u_n$ s.t

$1 = \|u_n\| > \frac{n}{h_n}\|f_n\| + n\|a^{1/2}u_n\|$

$\|f_n\| = o(h_n)$ $\qquad \|a^{1/2}u_n\| = o(1)$.

$b \in C_0^\infty$

$(Op(b)u_n, u_n)_{L^2} \to l(b)$

$\Rightarrow \forall\, b \in C_0^\infty \quad (Op(b)u_n, u_n)_{L^2} \to \langle \mu, b\rangle$ $\quad$ by diagonal procedure

i) $|\mu| = 1$

ii) $supp(\mu) \subset \{(x,\xi) : |\xi|^2 = 1\}$

$Op(b(-h_n^2|\xi|^2+1)u_n, u_n) = (Op(b)(h^2\Delta+1)u_n, u_n)$

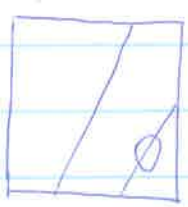
$\downarrow$
0

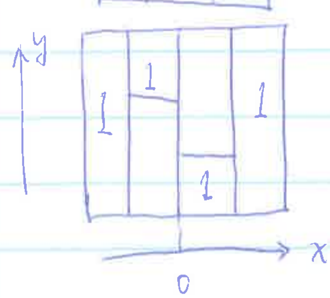
iii) $\xi \cdot \nabla\mu = 0$

$(\frac{1}{ih}[h^2\Delta + 1, Op(b)]\,u_n, u_n]$

$\Rightarrow$ "$\mu(x+s\xi, \xi) = \mu(x,\xi)$" , but $\|a^{1/2}u_n\| \to 0$

$\Rightarrow \langle \mu, a(x)\rangle = 0$.





1) Prop $\exists\, \varepsilon(h)$

$\|u_n\|_{L^2(|x| \leq h^{1/2}\varepsilon^{-1}(h))} \leq C\,\varepsilon(h)$

$h^{1/2}\varepsilon^{-2}(h) \le |x| << 1$

$a(x, y, hD_x, hD_y, \underbrace{\frac{\tilde{h}}{h}x}_{z}, \underbrace{\tilde{h}D_x}_{\xi})$

$\tilde{h} = h^{1/2}\varepsilon^{-1}(h)$

$\tilde{\mu}$ eqn on $\tilde{\mu}$