8/17/16 Tullia — Part (2)

Goals: (1) Extend Tullia's theorem on $S^n_= \alpha$ to boundaries of NCHS.

(2) QI rigidity of lattices in solvable Lie group.

Abelian $\subseteq$ Nilpotent $\subseteq$ Solvable.

$\mathbb{R} \times \mathbb{R}$ $\quad t \mapsto e^t$.

$(x,t)\cdot(x',t') = \cancel{e^t x x'}$

$ds^2 = dt^2 + e^{-2t} dx^2$. $\quad = (x + e^t x', t + t')$.

In general $N \rtimes_e \mathbb{R} = X_e$. $\quad \varphi_t = e^{tA}$.

e.g. $\mathbb{R}^2 \times \mathbb{R}$
$(x,y) \quad t$

Def^n Boundaries $\partial(N \rtimes_e \mathbb{R}) \simeq S^{n+1} = N \cup \{\infty\}$.

Pointed Sphere Conjecture (Cornulier/Xie).
Any self QI of X_e preserves $\{\infty\}$ when X_e is not symmetric space.

Def^n Bilip: $a\, d(x,y) \leq d(f(x), f(y)) \leq b\, d(x,y)$.
if $a = b$ — Similarity.

$G \subseteq \mathrm{bilip}(Y)$ is uniform.

Metrics on $\partial X_e \simeq N$ (N, d_e).

If $N = \mathbb{R}^n$, $\varphi_t = e^t I$.

If N cannot grp of φ can't dilation.

Then d_e is Carathéodory metric.

$\mathbb{R}^n \times \mathbb{R}$. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $d_A(\bar{v}, \bar{w}) = \max\{|\Delta x|, |\Delta y|^{\frac{1}{2}}\}$.

Tukia Type Theorem:

(D - Xie) If A diag. w/ pos. eigenvalues. and $G \subseteq \mathrm{bilip}(\mathbb{R}^n, d_A)$.

- G uniform ($\%_b < K$) & locally comp.
- G acts completely on distinct pts.
- G is amenable.

Then $\exists$ $f \in \mathrm{Bilip}(\mathbb{R}^n, d_A)$ s.t.

$$f G f^{-1} \subseteq \mathrm{Sim}(\mathbb{R}^n, d_A).$$

Proof: (D) adapt. Tukia so that up to conj.

If G acts by

$\gamma(x, y) = (a_\gamma(x + c_\gamma + h_\gamma(y))$

$$= \underset{\uparrow \text{sim}}{\sigma} \cdot (x + h\gamma, y)$$

Goal:- Conj. G by $g_0(x,y) = (x + h_0(y), y)$.
s.t $h_\gamma(y) = 0$, $\forall \gamma \in G$.

h_0 needs to be fixed pt. of (Affine) action G on
$E = \{ h : \mathbb{R} \to \mathbb{R} \mid 1/2\text{-holder}, h(0) \}$.

$$\gamma \longmapsto \pi_\gamma h + h_\gamma$$

$$\pi_\gamma h(y) = a_\gamma^{-1} \, h(a_\gamma^2 (y + d_\gamma)) - a_\gamma^{-1} h(a_\gamma^2 d_\gamma)$$

so $h(0) = 0$.

Def^n:- (Amenable)

G locally (amenable) compt top. group.

$G \circlearrowright K \subseteq E$ conts affine action on convex.

Compt subset of loc. con top sp.

then G has fixed pt.

QI Rigidity of lattices in solvable Lie grps.

Note:- X_φ does not have lattice.

but $X_{u_1, u_2} := N_1 \times N_2 \times_{(e_1, e_2^{-1})} \mathbb{R}$

e.g $SOL = \mathbb{R}^2 \times \mathbb{R}$ $\quad \varphi_t = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}$

Got two foliation by X_{u_1}, X_{u_2}.