Lecture 1

X = proper, geodesic metric space (thus complete)
CAT(0) inequality
Examples: Euclidean $\mathbb{E}^2$, $\mathbb{H}^2$, Trees, Buildings, products, gluings, etc.

Visual Boundary

$c', c : \mathbb{R}^+ \to X$ geodesic rays
$c \sim c' \leftrightarrow \exists D > 0$ s.t. $d(c(t), c'(t)) \leq D \quad \forall t \geq 0$
(asymptotic)

$\partial X = \{[c] \mid c \text{ a geod ray in } X\}$
$[c] := c(\infty)$, typical point $\xi \in \partial X$

Prop:
Convenient: Fix a basept. $p \in X$. Given $\xi \in \partial X$
$\exists$ unique ray $c : \mathbb{R}^+ \to X$ with $c(0) = p$, $c(\infty) = \xi$

$\partial_p X = \{\text{geod rays in } X \text{ based at } p\}$.

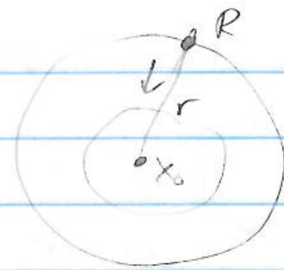
Cone Topology on $\partial_p X$. Let $c \in \partial_p X$. nbhd basis
$U(c, R, \varepsilon) = \{c' \in \partial_p X \mid d(c(R), c'(R)) < \varepsilon\}$ cmpt since X proper
(actually can topologize $\overline{X} = X \cup \partial_p X$ this way).
FACT: If we change basept, we get a homeo. topology

FACT: Given any isometry γ of X, $\overline{\gamma}$ is a homeo of this topology.
(Inverse limit of metric balls centered at p gives a topology on $\overline{X} = X \cup \partial X$.)

② (1.5)

Fancy definition

Fix $x_0 \in X$ $\qquad \overline{B(x_0,r)}$



unique projection

Consider the inverse limit $\varprojlim \overline{B(x_0,r)} \subseteq \{\text{maps } [0,\infty)\to X\}$

each point is a map $c:[0,\infty)\to X$ s.t. if $r'\geq r$ then $p_r(c(r')) = c(r)$. Two types

- either $c(r')\neq c(r)$ $\forall r\neq r'$ (so that c is a ray eman. from x_0)

or $\exists\, r_0$ s.t. $c(r') = c(r_0)$ $\forall r'\geq r_0$

$c:[0,r_0]\to X$ geodesic from x_0 to $c(r_0)$

and $c|_{[r_0,\infty)}$ is constant at $c(r_0)$.

$\overline{X} = X\cup\partial X$ $\qquad \exists\ \varphi(x_0): \overline{X}\to \varprojlim \overline{B(x_0,r)}$ bijection

If $x\in X$, then we can assign the map $[x_0,x]\cup$ const map

If $\xi\in\partial X$ then " the geod ray c from x_0 to $c(\infty)=\xi$

~~Topologize~~ Let $\mathcal{T}(x_0)$ be the topology that makes this $\varphi(x_0)$ a homeo.

- $i: X\to\overline{X}$ gives a homeo from X onto a dense open subset of $\overline{X}$
- $\overline{X}$ is cmpt when X is proper.
- ∂X is a closed subspace & therefore cmpt.

(We write $\partial_\infty X$ to be the subspace topology on ∂X)

But of course this is not always helpful for understanding examples...

Examples:

(1) $X = \mathbb{E}^2$ (or $\mathbb{E}^n$). $\partial_\infty X \cong S^1$ ($\bar{X} = X \cup \partial X$ homeo. closed Ball)

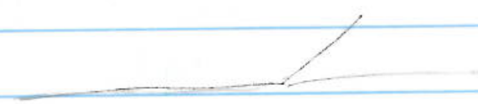
(2) $X = \mathbb{H}^2$ (or $\mathbb{H}^n$, $n \geq 2$) $\partial_\infty X \cong S^1$

Disk model $\mathbb{H}^2$ as poincare [illegible] on open Disk.

Lemma: 2 geod. rays c, c' are asymptotic $\Leftrightarrow$ they converge to the same pt. of the sphere of radius 1 in this model.

Each geod ray is a circle orthogonal to the unit sphere. The closure of this arc has one end pt. on the sphere of radius 1

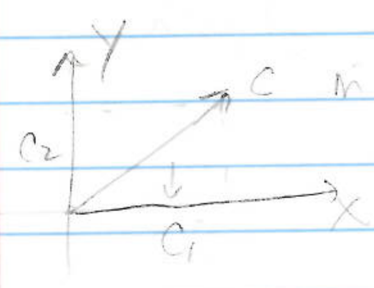
(3) $X = T$ locally finite infinite tree.



$\partial_\infty X \cong$ Cantor set

exercise: to prove this directly by showing that the topology satisfies the nec properties.

(4) $\partial_\infty (X \times Y) \cong \partial_\infty X * \partial_\infty Y$. spherical join



(2'). $\partial_\infty \tilde{M} \cong S^{n-1}$ if $\tilde{M}$ = univ. cover of Riem cmpt n-mnfld of NPC.

7

Angles.

Given geodesics $c, c' : I \to X$ with $c(0) = c'(0) = p$

$$\angle_p(c, c') = \lim_{t \to 0} \angle_p(c(t), c'(t))$$

comparison angles

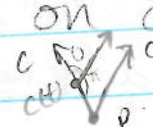
Thm: For geodesic rays $c, c' : \mathbb{R}^+ \to X$ $c(0) = c'(0) = p$ this can also be defined as:

$$\lim_{t \to \infty} 2 \cdot \arcsin\left(\frac{d(c(t), c'(t))}{2} \right).$$

Defn: For $\xi, \xi' \in \partial X$ define the angle btwn them as

$$\angle(\xi, \xi') = \sup_{p \in X} \left\{ \angle_p(c, c') \;\middle|\; \begin{array}{l} c(\infty) = \xi \\ c'(\infty) = \xi' \end{array} \right\}$$

FACT: This is a metric on ∂X. called the angle metric.



$c \neq c'$, then $\exists t$ s.t. $c(t) = c'(t)$

If $\angle_{c(t)}(c, c'')_1 \neq 0$, then we get another geod.

ex: $(\partial \mathbb{E}^2, \angle) \equiv$ unit circle in $\mathbb{E}^2$ but $(\partial \mathbb{H}^2, \angle)$ discrete.

So this metric topology on ∂X distinguishes them.

Tits Metric on ∂X

Recall, a metric space (X, d) is called a length space if $\forall p, q \in X$, $d_X(p, q) = \inf\{\ell(\text{rect. paths btwn them})\}$

where by length we mean: $c : [a, b] \to X$ cont.

$$\ell(c) = \inf_{\substack{\text{all partitions} \\ a = t_0 \leq t_1 \leq \cdots \leq t_n = b}} \left\{ \sum_{i=1}^{n-1} d(c(i), c(i+1)) \right\}$$

This is either finite or ∞

when finite, it is rectifiable

Given any metric space (X, d) one can obtain the associate length metric $\bar{d}$ on X as follows:

$$\bar{d}(p,q) = \inf\{\ell(\text{rect paths})\} \text{ or } \infty \text{ if none exist.}$$

The Tits metric on ∂X is defined to be the length metric Td assoc. to $(\partial X, \angle)$.

$\partial_T X = (\partial X, Td)$

exs: 5 quarter planes, $\mathbb{H}^2$, $\mathbb{E}^2$

$\angle(p,q) = Td(p,q)$ in $\mathbb{E}^2$

$\forall p \neq q$ $\angle(p,q) = \pi$, but $Td(p,q) = \infty$ in $\mathbb{H}^2$

- $T \times \mathbb{R}$ vs $\mathbb{H}^2 \times \mathbb{R}$

Here, the Tits metrics are the same but the visual bdries are different!!

- Favorite example
- $\pi_1(\text{Fig } 8)$ versus π_1(Hyp. 3 mfld with totally geod. bdry)

Fund Fact:

The map $id: \partial X \to \partial X$ induces a cont. bijection $\partial_T X \to \partial_\infty X$ that is definitely not a homeo in general.

The other 3 facts.

(4) $X = T \times \mathbb{R}$, $\partial_\infty X \cong \Sigma C$ suspension of Cantor set

Q: $\mathbb{E}^2$ & $\mathbb{H}^2$ have same visual bdry but are really different so how do we distinguish them at ∞
A: Another Topology.

hyp 3 mnfld with one torus cusp
(5) π_1 (Fig 8 knot group) ? maybe. we'll see
Sierpinski carpet.

(6). π_1 (hyp 3 mnfld with totally geod. bdry) - also Sierpinski carpet maybe.

(7) Group Theory: $G = \underset{\langle x \rangle}{\mathbb{Z}} \oplus \underset{\langle y \rangle}{\mathbb{Z}} *_{\underset{\langle t \rangle}{\mathbb{Z}}} = \langle x, y, t \mid [x,y] = 1,\ t^{-1} x t = y \rangle$

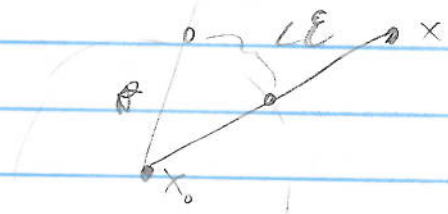
Consider X = Presentation 2 complex for ↗.
This is a CAT(0) cube complex.
$\partial_\infty X$ is really crazy.. I cannot tell you what it is up to homeo but I can tell you some properties that is has: conn (grp is 1-ended)
not. loc connected
not (globally) path conn.

5

A nbhd basis for this topology on $\bar{X} = X \cup \partial X$ can be given by the metric balls in X along with following sets that are basic nbhds of pts. at ∞. Let $\xi \in \partial X$, param. by a geod. c, $c(\infty) = \xi$.

$$U(c, r, \varepsilon) = \{ x \in \bar{X} \mid d(x, c(0)) > r,\ d(pr(x), c(r)) < \varepsilon \}$$

$c(\infty) = \xi$



- Changing basept does not affect the topology. This requires proof but is not difficult using the inverse limit definition.

- Any isometry of X extends to a homeo of $\bar{X} = X \cup \partial X$ - in particular, a homeo of ∂X.

<u>examples</u>: $\partial_\infty \mathbb{E}^2 \cong S^1$, $\partial_\infty \mathbb{H}^2 \cong S^1$, M - Riem. mnfld of NPC. (cmpt)
$\tilde{M}$ = univ. cover $\to \partial_\infty \tilde{M} = S^{n-1}$. ($\partial_\infty T$ = cantor set) $\partial_\infty (X \times Y) = \partial_\infty X * \partial_\infty Y$ (join)

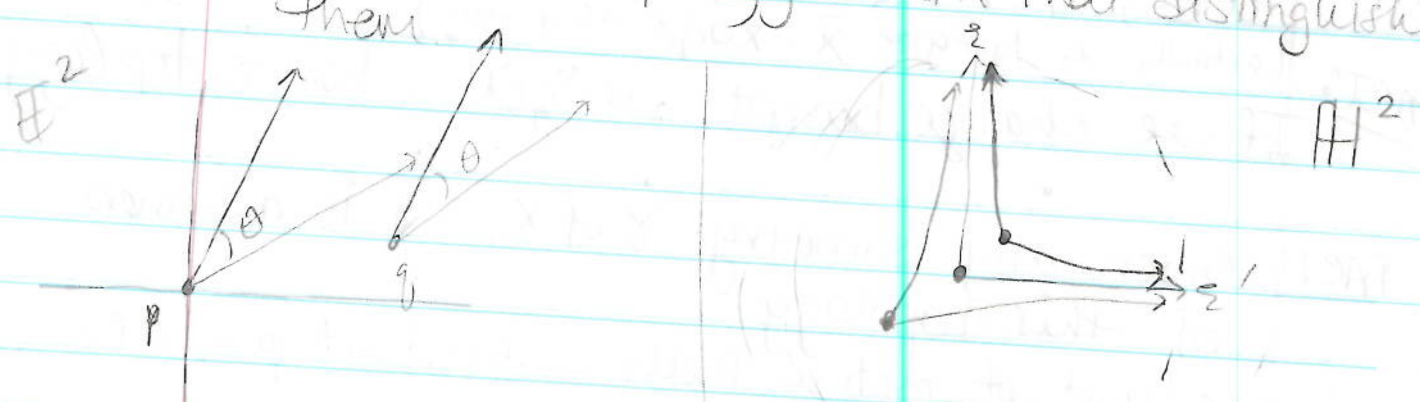
Thm: (Davis / Januszciewicz) produce examples X of contractible topological mnflds of dim $n \geq 5$ which support a (CAT(0)) metric on X with a group $G \curvearrowright X$ and $\partial_\infty X$ not a sphere.

<u>ex</u>: $X = T \times \mathbb{R} \to \partial_\infty X = \Sigma C$

<u>Remark</u>: any isom γ of X would induce a homeo $\bar{\gamma}$ of $\partial_\infty X$. Since the susp. pts. of $\partial_\infty X$ are top. distinguished, $\bar{\gamma}$ would have to leave them fixed or interchange them... keep this in mind for later.

<u>Q</u>: We saw that $\mathbb{E}^2$ & $\mathbb{H}^2$ have the same visual boundary but we know they are really different geometries - how can we see this at infinity?

<u>A</u>: Put another topology on ∂X that distinguishes them.



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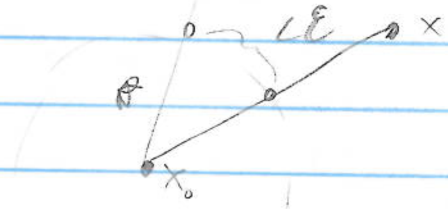

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