4:15

Lecture 2: Using the Boundary

We have seen that the Tits and visual topologies can be used to distinguish geometries from each other. Let's add some group theory now.
Before I do, 3 more geometric results that are useful:
The results hold in varying degrees of generality - I will use the strongest assumption: X admits a geometric group action by a group G

Bridson/Haefliger (1) If $\partial_T X$ is discrete, then X hyperbolic
Bosché (2) If $\partial_T X \cong \partial_\infty X$ (~~and X is geod complete~~), then X is flat and both are spheres.
Caprace/Monod (3) If $\partial_T X$ splits as a join, then $X \equiv Y \times Z$.

When we have $G \curvearrowright X$ geometrically, we get that each $g \in G$ extends to $\bar{g}: \partial_\infty X \to \partial_\infty X$ a homeo and $\tilde{g}: \partial_T X \to \partial_T X$ an isometry.
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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Noureen Khan **Email/Phone:** noureen.khan@unt.edu/ 2142842214

Speaker's Name: Kim Ruane

Talk Title: Visual and Tits Boundaries of CAT(0) spaces

Date: 8/17/2016 / / **Time:** 4:15 p am / pm (circle one)

List 6-12 key words for the talk: Hyperbolic spaces, Euclidean spaces
Geodesic Metric space, Locally finite trees, Tits boundaries, visual boundaries

Please summarize the lecture in 5 or fewer sentences: Part-2
1. Continued with more examples (fundamental group) from lecture 1.
2. Facts and results related to CAT(0) were presented.
3. Cited and presented the theorem about fundamental closed 3 manifold and CAT(0) groups.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

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(2)

(2) Sps $G \curvearrowright \mathbb{E}^2$, then $\exists\, g \in G$ of ~~∞~~ order s.t. g central in a subgrp of finite index and $G/\langle g \rangle$ 2-ended.

- That $\exists$ a $g \in G$ of infinite order is easy for $\mathbb{E}^2$ but is also true for all CAT(0) groups by E. Swenson.

$\exists\, A_g$ an axis for g inside $\mathbb{E}^2$...
Consider $Min(g)$ – could be a line (glide reflection)
→ the entire plane. (translation)

$P(g) = \{$lines in $\mathbb{E}^2$ that are parallel to $A_g\}$.
In general, $P(g)$ is larger than $Min(g)$
which can cause problems...

ex: Construct an example of a group $G \curvearrowright X$ CAT(0)
$g \in G$, ~~$Min(g) = T \times \mathbb{R}$~~
$Min(g) = $ line, $P(g) = T \times \mathbb{R}$,
$Min(g^p) = F \times \mathbb{R}$. $\subseteq Min(g^{p+1}) \subseteq \dots$
finite subtree (larger finite subtree $\times \mathbb{R}$)

(never stabilizing...)

Fundamental Facts needed

$|g| = \inf\{d(x, g\cdot x) \mid x \in X\}$ $\quad$ $Min(g) = \{x \in X \mid d(x, g\cdot x) = |g|\}$

(0) ~~If $g \in G \curvearrowright X$, then g is a "hyperbolic"~~ a

(1) $g \in Isom(X)$, one of three types

- elliptic - $Min(g) \neq \emptyset$ and $|g| = 0$. (fixed pt)
- hyperbolic - $Min(g) \neq \emptyset$ and $|g| > 0$
- parabolic $Min(g) = \emptyset$ and $|g| = 0$

FACT: If $G \curvearrowright X$ geom, then we can rule out parabolics.

(We can ignore the elliptics for now)

(2) If $g \in G$ infinite order, then $\exists$ geod

- line $A_g : \mathbb{R} \to X$ on which g acts by translation (called an axis for g)

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- The set of all axes is $Min(g)$ and this set decomposes as $Min(g) = Y \times \mathbb{R}$ for some closed, convex $Y \subseteq X$.

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$Fix(\bar{g}) = \partial_\infty Min(g)$ and in fact

(4) Kleiner/Kapovich (Bridson–Mosher) independently

Sps $G = \pi_1(M)$ ← closed 3-mnfld, $\tilde{M}$ = univ. cover with a G-inv. CAT(0) metric

Then either G is hyperbolic or G contains $\mathbb{Z} \oplus \mathbb{Z}$.

- Conjecture for (all) CAT(0) groups:

(actually they prove this for any 3-dim'l P.D. group over a certain type of ring (comm. hereditary))

Weak Hyperbolization: A class of groups satisfies WH if every group in the class is either δ-hyp or contains $\mathbb{Z} \oplus \mathbb{Z}$.

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