Mapping class groups and $Out(F_n)$

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The three great families of groups

The simplest interesting topological space is S^1. What is the second?

The three great families of groups

The simplest interesting topological space is S^1. What is the second? Three answers:

- $T^n = (S^1)^n \quad \to \quad SL_n(\mathbb{Z})$ and arithmetic groups.
- surfaces $\Sigma \quad \to \quad$ mapping class group $Mod(\Sigma)$.
- graphs $\quad \to \quad Out(F_n)$.

Up to index 2, $Out(F_2) = Mod(T^2) = SL_2(\mathbb{Z})$.

Definitions, for the record

$$Mod(\Sigma) = Homeo_+(\Sigma)/isotopy = \pi_0(Homeo_+(\Sigma))$$

$$Out(F_n) = Aut(F_n)/InnerAutos$$

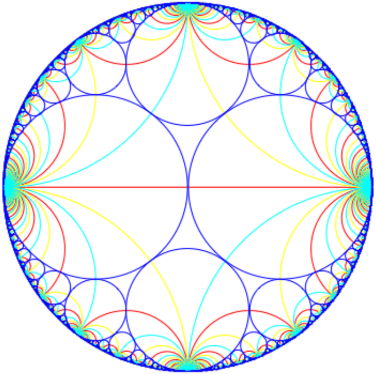
Mantra: $Out(F_n)$ is just like $Mod(\Sigma)$

$SL_2(\mathbb{Z})$

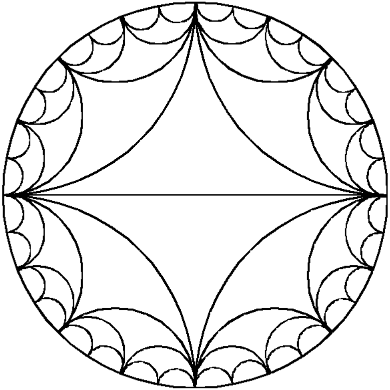
$SL_2(\mathbb{Z})$ acts by isometries on hyperbolic plane

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

$SL_2(\mathbb{Z})$



action cocompact on $\mathbb{H}^2 \setminus$ horoballs

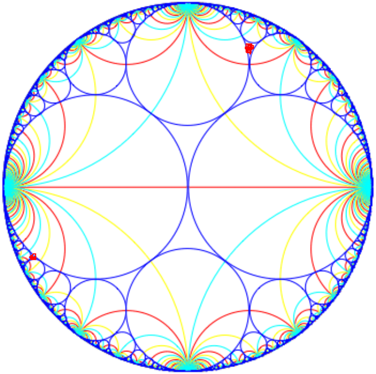


Farey graph F is the incidence graph; it is a quasi-tree.

There is a natural coarse Lipschitz map $\mathbb{H}^2 \to F$.
Geodesics project to quasi-geodesics

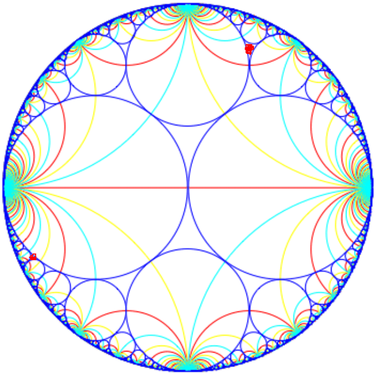
Distance formula in $SL_2(\mathbb{Z})$

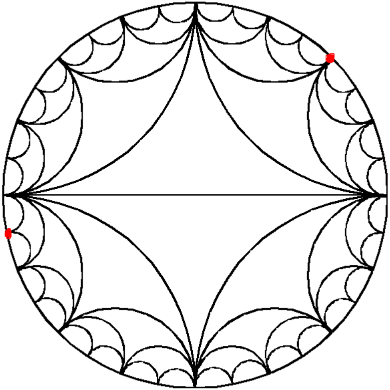
Say we want to estimate the distance between two points x, y in $SL_2(\mathbb{Z})$, coarsely identified with $\mathbb{H}^2 \setminus$ horoballs.



Distance formula in $SL_2(\mathbb{Z})$

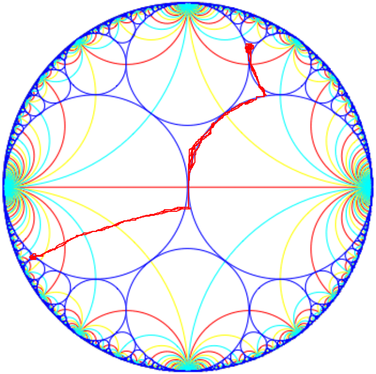
A lower bound is the distance $d_F(x, y)$ between projections to F.





Distance formula in $SL_2(\mathbb{Z})$

As an upper bound, this fails badly, e.g. when the two points are along the same horoball. To fix it, consider projections to boundaries of horoballs.



$d_H(x, y)$ is the distance along $\partial H = \mathbb{R}$ (path metric) between the projections of x, y.

Distance formula in $SL_2(\mathbb{Z})$

$$d(x,y) \asymp d_F(x,y) + \sum_{H, d_H(x,y)>6} d_H(x,y)$$

Special case of the Masur-Minsky distance formula for $Mod(\Sigma)$.

$SL_2(\mathbb{Z})$	$Mod(\Sigma)$	$Out(F_n)$
$\mathbb{H}^2$	Teichmüller space	Culler-Vogtmann's Outer space CV_n
point in $\mathbb{H}^2$	marked hyperbolic surface marked Riemann surface	marked metric graph free simplicial F_n-tree weighted sphere system
point in $\partial\mathbb{H}^2$	measured geodesic lamination	$F_n - \mathbb{R}$-tree
geodesic in $\mathbb{H}^2$	Teichmüller geodesic	folding path surgery path
hyperbolic metric	Teichmüller metric	Lipschitz metric
horoball	where a curve is short	where a free factor is small
Farey graph	curve complex $\mathcal{C}(\Sigma)$	free factor complex $\mathcal{FF}_n$ free splitting complex $\mathcal{FS}_n$
loxodr. isometry	pseudo-Anosov	fully irreducible auto
horoball projections	subsurface projections	subfactor projections

Similarities

- $Teich(\Sigma)$, CV_n contractible. Both groups have virtually finite $K(\pi, 1)$, *vcd* computed.
- homological stability, stable homology, duality
- Tits alternative, solvable groups are virtually abelian
- classification of automorphisms
- finitely many growth rates, all algebraic integers
- $\mathcal{C}(\Sigma), \mathcal{FF}_n, \mathcal{FS}_n$ hyperbolic

Differences

- $Out(F_n)$ has elements that grow polynomially but not linearly
- For $f \in Out(F_n)$ the growth rates of f and f^{-1} are typically different
- $Out(F_n)$ contains Kolchin subgroups that are not virtually abelian but contain only polynomially growing elements. These are also distorted and are not detected by subfactor projections.
- $Out(F_n)$ has an exponential Dehn function; $Mod(\Sigma)$ quadratic
- $Out(F_n)$ has no (known?) distance formula

Questions

- Is there a finite index subgroup of $Mod(\Sigma)$ or $Out(F_n)$ that maps onto $\mathbb{Z}$? Onto F_2?
- Does $Mod(\Sigma)$ or $Out(F_n)$ have property (T)?
- Does $Mod(\Sigma)$ or $Out(F_n)$ admit an action on a $CAT(0)$ cube complex with unbounded orbits?

convex cocompactness

A finitely generated subgroup $Q \subset Mod(\Sigma)$ is convex cocompact if the associated extension

$$1 \to \pi_1(\Sigma) \to G \to Q \to 1$$

is δ-hyperbolic; equivalently, the orbit map $Q \to \mathcal{C}(\Sigma)$ is a QI embedding.

- Are f.g. subgroups Q consisting of 1 and pseudo-Anosov mapping classes convex cocompact?
- Is every convex cocompact subgroup virtually free?

Large scale geometry of $Out(F_n)$

- Is there a version of the distance formula for $Out(F_n)$?
- For which elements $f \in Out(F_n)$ is there an action of $Out(F_n)$ [or a finite index subgroup] on a hyperbolic space with f loxodromic?
- Prove QI rigidity for $Out(F_n)$.
- Does $Out(F_n)$ have finite asymptotic dimension? Does $\mathcal{FF}_n$ or $\mathcal{FS}_n$ have finite asymptotic dimension?
- Is $Out(F_n)$ boundary amenable?