HYPERBOLIC GROUP EXTENSIONS

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ABSTRACT. William Thurston's seminal construction of a hyperbolic 3-manifold fibering over the circle gave the first example of a Gromov hyperbolic surface-by-cyclic group. This breakthrough sparked a flurry of activity, and there has subsequently been much progress towards developing a general theory of hyperbolic group extensions. In this talk I will review some of this basic theory – including combination theorems for ensuring a group extension is hyperbolic and structural theorems about general hyperbolic extensions – and then discuss my work with Sam Taylor studying hyperbolicity in the specific context of free group extensions. For instance, we use the geometry of Outer space to show that every purely atoroidal subgroup of $\mathrm{Out}(F_n)$ that quasi-isometrically embeds into the free factor complex gives rise to a hyperbolic extension of F_n.

K, Q groups, an extension of K by Q is a group G fitting SES

$$1 \to K \to G \to Q \to 1.$$

Basic question. If K and Q are word-hyperbolic, when will G be hyperbolic?

e.g. [stupid extension]

$$1 \to K \to K \times Q \to Q \to 1$$

never hyperbolic provided $|K|, |Q| = \infty$.

$1 \to K \to G \to Q \to 1 \rightsquigarrow$ homomorphism $Q \to \mathrm{Out}(K)$ "outer action"

$q \mapsto$ conjugation by any lift $t_q \in G$.

For G hyperbolic, need $\ker(Q \to \mathrm{Out}(K))$ finite, so in particular we need $\mathrm{Out}(K)$ infinite.

Date: 22 August 2016.

JSJ decomposition for hyperbolic groups (Sela, Rips–Sela 1997): If G hyperbolic (and K torsion-free), K must be a free product of surface and free groups.

First example. Σ closed surface, $f : \Sigma \to \Sigma$ homeomorphism $\rightsquigarrow$ mapping torus $M_f = \Sigma \times [0,1]/(x,1) \sim (f(x),0)$.

$$1 \to \pi_1(\Sigma) \to \pi_1(M_f) \to \mathbb{Z} \to 1$$

(Jørgensen, 1977) explicit example of f such that M_f admits hyperbolic Riemannian metric.

(Thurston) f pseudo-Anosov $\implies$ M_f hyperbolic 3-manifold. Moreover, M_f hyperbolic $\implies$ $\pi_1(M_f)$ hyperbolic $\implies$ f pseudo-Anosov (otherwise curve fixed by power of $f \rightsquigarrow \mathbb{Z} \times \mathbb{Z}$ subgroup).

Bestvina–Feighn Combination Theorem 1992

Consequence: If K hyperbolic and $\phi \in Aut(K)$ hyperbolic (i.e. $\exists\, \lambda > 1, \exists\, m \in \mathbb{N}$ such that $\forall\, g, \lambda|g| \leq \max\{|\phi^m(g)|, |\phi^{-m}(g)|\}$) then $K \rtimes_\phi \mathbb{Z}$ is hyperbolic.

Second example. $1 \to F_n \to F_n \rtimes_\phi \mathbb{Z} \to \mathbb{Z} \to 1$ where $\phi \in \mathrm{Out}(F_n)$ *atoroidal*.

Third examples.

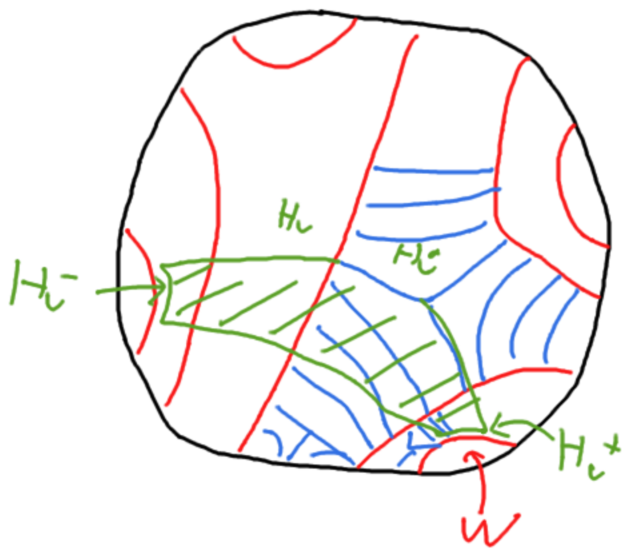
$$1 \to \pi_1(\Sigma) \to G \to F_k \to 1 \quad \text{Mosher 1997}$$

$$1 \to F_n \to G \to F_k \to 1 \quad \text{Bestvina–Feighn–Handel 1997}$$

Graph of spaces: X finite cell complex, $X \xrightarrow{p} \Gamma$ finite graph, edge spaces X_e : p^{-1}(midpoints of edges), vertex spaces X_v : components of $X \setminus \cup_e X_e$.

Universal cover $\tilde{X} \to T$, tree of spaces $\tilde{X}_v, \tilde{X}_e$.

Hallway Δ is



- λ*-hyperbolic* if $\lambda l(\Delta_0) \leq \max\{l(\Delta_{-m}), l(\Delta_m)\}$.
- ρ-thin if $d_{\tilde{X}_v(j)}(\Delta(j,t), \Delta(j+1,t)) \leq \rho$
- essential if edge path $e(-m), \ldots, e(m)$ non-backtracking.

Theorem (BF). *Suppose vertex spaces X_v hyperbolic, edge spaces $\hat{X}_e$ qi-embedded in vertex spaces.*

$\exists\, \lambda, m \in \mathbb{N}$ such that $\forall\, \rho$, there is a threshold $T(\rho)$ such that every ρ-thin essential hallway length $2m$ and $l(\Delta_0) \geq T(\rho)$ is λ-hyperbolic.

Then $\tilde{X}$ is hyperbolic.

Sketch. $\tilde{X}$ hyperbolic $\iff$ linear isoperimetric inequality $\iff$ (by Gromov) subquadratic isoperimetric inequality.

Let $C : S^1 \to \tilde{X}$ be a loop, $\pitchfork \bigcup_e \tilde{X}_e$, extend to disk $\Delta : D^2 \to \tilde{X}$, assume

- $W = \Delta^{-1}(\cup_e \tilde{X}_e)$ properly embedded arcs
- $len(W)$ minimal
- each component P of $D^2 \setminus W$, $\Delta(P)$ quasi-geodesic polygon with area $\leq A(l(\partial P))$.

Each P looks like a tree.

Singular fibers decompose W into set $\mathcal{V}$ of segments. Use fibration to build hallway H_v about each $V \in \mathcal{V}$ that has

- length $2m$ (assume all 3-hyperbolic), or
- hits S^1

3-hyperbolic: $l(H_v^0) \leq \frac{1}{3}\max\{l(H_v^-), l(H_v^+)\}$.

Prove theorem:

$$area(\Delta) = \sum_P area(P) \leq A(l(c) + 2l(W))$$

Must bound $l(W)$ by $l(c)$:

$$l(W) = \sum_{v \in V} l(H_v^0) \leq Km(l(c)) + \sum_v \frac{1}{3}(l(H_v^-) + l(H_v^+))$$
$$\leq Km(l(c)) + \frac{2}{3}l(W)$$

General Theorem of Hyperbolic Extensions?

- For $\pi_1(\Sigma)$-extensions, hyperbolic $\iff$ compact subgroups of mapping class group.
- For F_n? Each $\Gamma \leq \mathrm{Out}(F_n) \rightsquigarrow$ extension

  $$1 \to F_n \to E_\Gamma \to \Gamma \to 1$$

  where we take pre-image of $\Gamma \leq \mathrm{Out}(F_n)$ corresponding to

  $$1 \to F_n \to Aut(F_n) \to \mathrm{Out}(F_n) \to 1$$

  Question: when is E_Γ hyperbolic?

Definition (co-surface graph). CS simplicial graph with

- vertices: primitive conjugacy classes in F_n

- edges: $\alpha - \beta \iff \exists$ once punctured surface Σ, $\pi_1(\Sigma) \cong F_n$ such that α and β both correspond to *simple* closed curves on Σ.

CS is an ∞-diameter hyperbolic graph.

$\phi \in \mathrm{Out}(F_n)$ acts loxodromically $\iff$ atoroidal and fully irreducible.

There is a coarsely Lipschitz map $FF_n \to CS$.

Theorem (D.–Taylor)**.** *If* $\Gamma \leq \mathrm{Out}(F_n)$ *f.g. and orbit map* $\Gamma \hookrightarrow CS$ *is qi-embedding, then* E_Γ *is hyperbolic.*

Converse is false (take $\phi \in \mathrm{Out}(F_n)$ which is atoroidal but not fully irreducible, then $F_n \rtimes_\phi \mathbb{Z}$ is hyperbolic but embedding of cyclic subgroup generated by ϕ not qi-embedding).

Audience question: do you have examples where Γ is not virtually free? No.