SPECIAL CUBE COMPLEXES AND THE VIRTUAL HAKEN CONJECTURE

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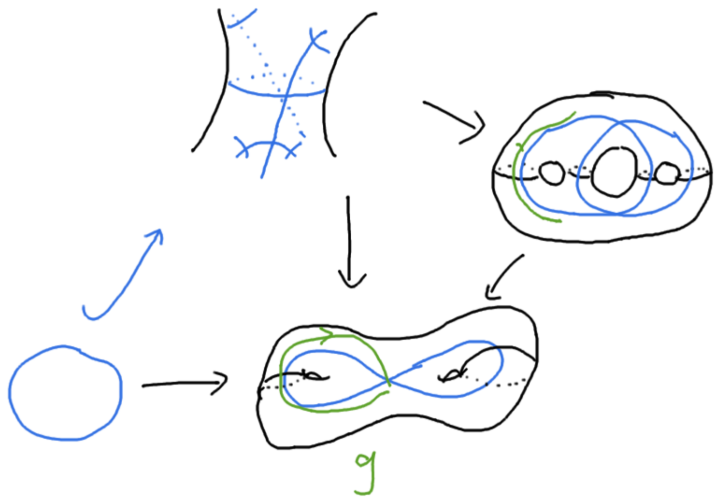
ABSTRACT. This will be an expository talk on the theory of special cube complexes and their application in resolving the virtual Haken conjecture.

1. SUBGROUP SEPARABILITY

Topological problem. Given $\Sigma^{n-1} \overset{\psi}{\looparrowright} M^n$ an immersion of compact manifolds, when does ψ lift to an embedding in some finite-sheeted cover $\hat{M} \to M$?

Necessary condition: if M_Σ is the cover corresponding to $\psi_*(\pi_1\Sigma)$, Σ should embed in M_Σ (probably ∞-sheeted).



What's the obstruction?

Group theoretically, we have $G = \pi_1 M, H = \pi_1\Sigma$, some $g \in G \setminus H$. Want $\hat{G} <_{\text{f.i.}} G$ so $H < \hat{G}$ but $g \notin \hat{G}$.

Date: 23 August 2016.

Definition. $H < G$ is *separable* if for all $g \in G \setminus H$, there is some $\hat{G} <_{\text{f.i.}} G$ so $H < \hat{G}$ but $g \notin \hat{G}$.

Equivalently

(1) $H = \cap \left\{ \hat{G} <_{\text{f.i.}} G \,\middle|\, H < \hat{G} \right\}$

(2) H closed in profinite topology on G

(3) Scott Criterion: if $\pi_1(K) = G$, and $K_H \to K$ is the cover corresponding to H, and $C \subset K_H$ any compact set, then C embeds in some intermediate finite-sheeted cover.

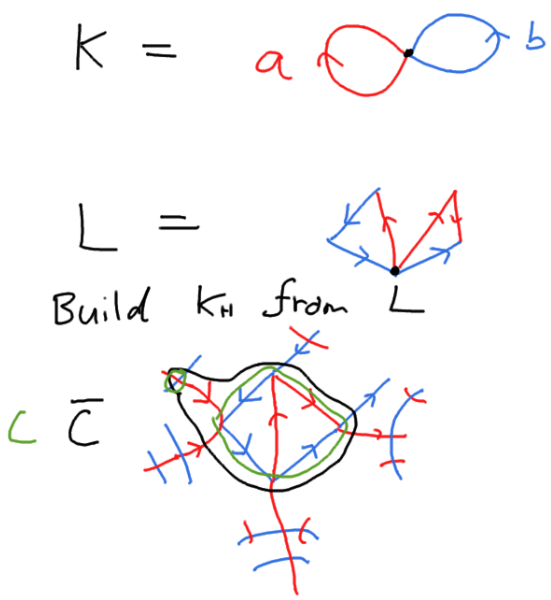
Definition. G is *LERF* if every f.g. subgroup is separable.

Theorem. *Free groups are LERF.*

Proof by example (c.f. Stallings)

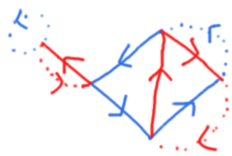
$G = \langle a, b \rangle$ free.

$H = \langle ab^2, a^2b^{-1} \rangle$



Build finite-sheeted cover

$\hat{K}$



$\overline{C}$ = smallest connected subgraph containing C.

"Canonical" completion of $\overline{C}$, the connected thing containing C.

Also a retraction $\hat{K} \to \overline{C}$

We showed

(1) Scott's criterion holds, but also

(2) H is a virtual retract of G

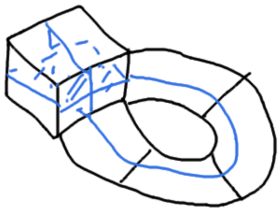
2. Special cube complexes

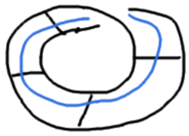
Would like to generalize to non-positively curved cube complexes (NPCCC's). NPC means locally CAT(0).

Wise, Burger–Mozes: ∃ NPC square complexes with no finite-sheeted covers. (Indeed, Bridson–Wilton show that you cannot decide whether a NPC square complex has a finite-sheeted cover or not.)

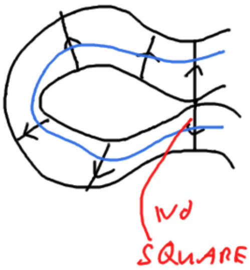
Non-definition of special cube complexes: those NPC cube complexes which admit (canonical) completions and retractions to locally isometrically immersed subcomplexes.

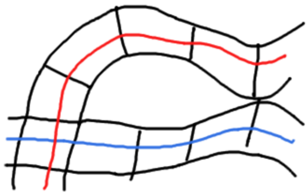
Definition. (1) A *special cube complex* is a NPCCC without certain hyperplane pathologies: self-intersection, one-sided,





self-osculate, and inter-osculate.





(2) Special cube complexes are those which admit a locally isometric immersion to $S(\Gamma)$ (Salvetti complex for a RAAG).

So $\pi_1($ special CC $)$ is $\mathbb{Z}$-linear, residually finite, RFRS, geometrically nice subgroups are separable (which are all virtual properties, i.e. commensurability invariants).

Definition. G is (compactly) *cubulated* if $G = \pi_1(X)$ for X compact NPCCC. It is *special* if moreover X is special. G is *virtually special* if $\exists\, \hat{G} <_{\text{f.i.}} G$ so $\hat{G}$ is special.

Theorem (Haglund). *G hyperbolic and virtually special, then QC subgroups of G are separable (G is QCERF).*

Theorem (Agol's theorem). *G hyperbolic and cubulated $\implies$ G is virtually special. (Conjectured by Wise.)*

3. 3-MANIFOLDS

(Assume compact and orientable.)

M^3 is *irreducible* if every 2-sphere bounds a ball.

An irreducible 3-manifold is *Haken* if $\exists\, \Sigma^2 \hookrightarrow M$ 2-sided, π_1-injective, and $\chi \leq 0$.

Example. $f : \Sigma \to \Sigma$ a homeomorphism, $\chi(\Sigma) \leq 0$. Then

$$M_f = \Sigma \times [0,1] / \text{ glue ends by } f$$

is Haken (in fact *fibered*).

Haken manifolds have *hierarchies* (cut along essential surfaces repeatedly, ending with $\sqcup B^3$).

Get pre-geometrization proofs (mostly by Waldhausen) of

(1) $\hat{M} \cong \mathbb{R}^3$

(2) solve word problem in π_1

(3) homeomorphism problem

(4) geometrization (Thurston)

Question (Waldhausen 1968)**.** Does irreducible $\implies$ virtually Haken?

After Perelman, question reduces to hyperbolic M.

Kahn–Markovic (2009): M closed hyperbolic $\implies$ $\pi_1(M)$ has lots of quasi-convex surface subgroups.

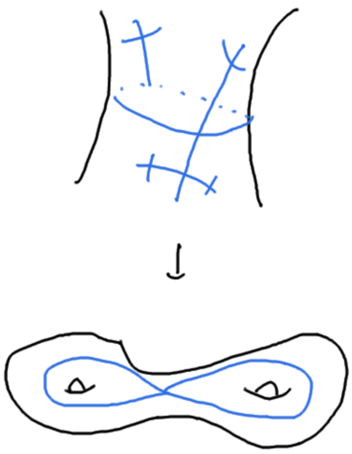
So if any are separable, get virtual Haken.

4. Codimension one subgroups (Sageev construction)

Definition. $H < G$ both f.g., Γ a Cayley graph for G.

$e(G, H) = \#$ ends of $H \backslash \Gamma$

H is *codimension one* if $e(G, H) > 1$.



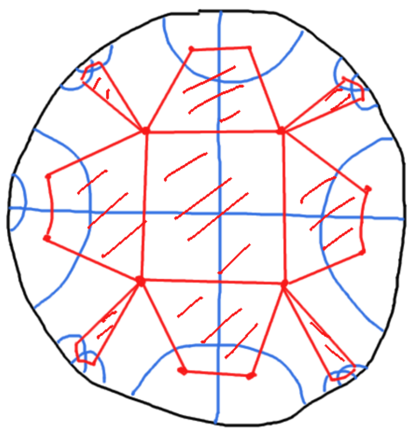
Example. $\Sigma^{n-1} \overset{\psi}{\looparrowright} M^n$ 2-sided, π_1-injective. Then $\psi_*(\pi_1\Sigma)$ is codimension one subgroup.

Example. $G = A *_C B$ or $A*_C$, then C is codimension one.

Sageev construction: Given $\mathcal{H} = \{H_1, \ldots, H_n\}$ of codimension one subgroups, get a CAT(0) cube complex X on which G acts.

Example.

If $\Sigma \hookrightarrow M$ get Bass–Serre tree.



Here $G \curvearrowright X$ is cocompact but not proper, so we haven't cubulated $G = \pi_1 M$.

Cocompactness criterion: If G hyperbolic and each $H \in \mathcal{H}$ is QC, then $G \curvearrowright X$ cocompactly.

Properness: G hyperbolic, $\mathcal{H}$ codimension one, QC. If every $p, q \in \partial G$ are separated by some $\Lambda(gH)$, then the action is proper.

What does "lots of" from Kahn–Markovic mean? Enough to get a proper action.

Corollary. *If M^3 is closed hyperbolic, $\pi_1 M$ is cubulated.*

Other examples of "cubulated" groups.

Mixed 3-manifold groups and many graph manifold groups.

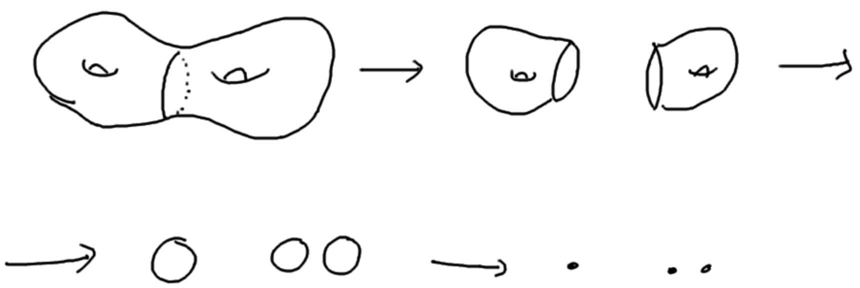
Many free-by-cyclic groups (including hyperbolic ones)

5. Some ideas in Agol's theorem

Wise's quasiconvex hierarchy theorem (QCH $\implies$ virtually special)

Definition. (1) Finite groups have a QCH of length 0.

(2) If G is hyperbolic and $\cong$ a graph of groups with QC edge groups and vertex groups with QC hierarchies of length $\leq n-1$, then G has QCH of length $\leq n$.



QHT $\Leftarrow$ malnormal combination theorems (Hsu–Wise, Haglund–Wise)

+ MSQT

6. Problems

See Wise and Agol's ICM papers from 2014 for lots of interesting problems (mostly unsolved).

Important problem. How far can these ingredients be pushed into the relatively hyperbolic / CAT(0) cube complex world?

Effectiveness / decidability:

Q1 Is virtual specialness decidable for compact cube complexes? c.f. Bridson–Wilton.

Q2 Can you predict the degree of a Haken cover of a 3-manifold, or a special cover of a cube complex with hyperbolic fundamental group? c.f. Patel, . . .

Meta-question: take your favorite question where you don't know the answer, and try to do it for cube complexes (possibly virtually special). For example: finite $K(G, 1)$ and no Baumslag–Solitar subgroups $\implies$ hyperbolic (audience comment: see work of Haglund for virtually special case).

Question of Wise: can you find a quasi-isometric pair of groups, one of which is virtually special and one of which has property (T).