THE POSET OF ACYLINDRICALLY HYPERBOLIC STRUCTURES ON A GROUP

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ABSTRACT. For every group G, we introduce the set of acylindrically hyperbolic structures on G, denoted $\mathcal{AH}(G)$. One can think of elements of $\mathcal{AH}(G)$ as cobounded acylindrical G-actions on hyperbolic spaces considered up to a natural equivalence. Elements of $\mathcal{AH}(G)$ can be ordered in a natural way according to the amount of information they provide about the group G. We will discuss some basic questions about the poset structure of $\mathcal{AH}(G)$ as well as more advanced results about the existence of maximal acylindrically hyperbolic structures and rigidity phenomena.

$G \curvearrowright S$ is *acylindrical* if $\forall\, \epsilon, \exists\, R, N$ such that $\forall\, x, y \in S$

$$d(x,y) \geqslant R \implies \#\{g \in G \mid d(x,gx) \leqslant \epsilon, d(y,gy) \leqslant \epsilon\} \leqslant N.$$

Example.

(0) $G \curvearrowright$ point (or bounded space)

(1) geometric $\implies$ acylindrical

(2) non-exceptional MCG $\curvearrowright$ curve complex.

Theorem-Definition. $\forall$ group G, the following conditions are equivalent:

(1) G admits an acylindrical non-elementary action on a hyperbolic space.

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(2) G admits a non-elementary action on a hyperbolic space with at least one loxodromic WPD element (definition of WPD given in Fujiwara's talk)

(3) $\exists$ a generating set X of G such that the corresponding Cayley graph $\Gamma(G, X)$ is hyperbolic, non-elementary ($|\partial\Gamma| > 2$) and $G \curvearrowright \Gamma(G, X)$ is acylindrical.

If G satisfies any of (1)-(3), it is called acylindrically hyperbolic.

Example.

(1) Non-elementary hyperbolic and relatively hyperbolic groups.

(2) Infinite $\mathrm{MCG}(\Sigma_{g,p})$

(3) $\mathrm{Out}(F_n), \geqslant 2$

(4) Most 3-manifold groups.

(5) Finitely presented groups of deficiency $\geqslant 2$.

(6) Directly indecomposable non-cyclic RAAGs

We want to measure / quantify the number of good generating sets in (3) or the number of actions as in (1) from the above Theorem-Definition.

$\mathcal{G}(G) := \{X \subseteq G \,|\, G = \langle X \rangle\}/\sim$, where $X \preceq Y$ if $\mathrm{id} : (G, d_Y) \to (G, d_X)$ is Lipschitz, and $X \sim Y$ if $X \preceq Y$ and $Y \preceq X$.

Example.

(1) All finite generating sets are equivalent.

(2) $X \subseteq Y \implies Y \preceq X$

$$\mathcal{AH}(G) := \{[X] \in \mathcal{G}(G) \,|\, \Gamma(G, X) \text{ is hyperbolic },\ G \curvearrowright \Gamma(G, X) \text{ is acylindrical } \}$$

$(\mathcal{AH}(G), \preceq)$ is called the poset of acylindrically hyperbolic structures on G (order induced by pre-order on generating sets).

Example. $\mathcal{AH}(G) \neq \varnothing, \forall\, G$ $([G] \in \mathcal{AH}(G))$

How large can the set of acylindrical hyperbolic structures be?

Theorem (Abbott–Balasubramanya–O.). *$\forall$ group G exactly one of the following holds:*

(1) $|\mathcal{AH}(G)| = 1$

(2) $|\mathcal{AH}(G)| = 2$. *In this case G is virtually cyclic.*

(3) G is acylindrically hyperbolic and $|\mathcal{AH}(G)| = \infty$. Moreover, in this case $\mathcal{AH}(G)$ contains chains and anti-chains of cardinality $2^{\aleph_0}$.

Idea.

(1) If G is acylindrically hyperbolic, then $\exists$ a hyperbolically embedded subgroup of G isomorphic to $F_2\times$ finite.

(2) (A.–B.–O.) If H is hyperbolically embedded in G, then $\mathcal{AH}(H)$ is a retract of $\mathcal{AH}(G)$ as a poset.

Definition. G is $\mathcal{AH}$-accessible (AHA) if $\mathcal{AH}(G)$ contains the largest element.

Example.

(1) $G = \ast_{n=2}^{\infty}(\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z})$ is *not* AHA.

(2) (Abbott) The Dunwoody inaccessible group is not AHA.

(3) If G is *not* acylindrically hyperbolic, then G *is* AHA.

Theorem (A.–B.–O.). *The following groups are AHA*

(1) Hyperbolic groups (trivial)

(2) Finitely generated relatively hyperbolic groups with $\mathcal{AH}$-accessible peripheral subgroups.

(3) $\mathrm{MCG}(\Sigma_{g,p})$

(4) 3-manifold groups

(5) RAAGs

Lemma (A.–B.–O.). *Suppose G acts acylindrically cocompactly on a hyperbolic graph Γ and assume that $|\mathcal{AH}(S)| = 1$ for all vertex stabilizers S. Then G is AHA.*

($G = \langle$finite set $\cup$ representatives of vertex stabilizers$\rangle$ gives X largest generating set)

"Corollary". There are at least 4 possible isomorphism classes of posets of $\mathcal{AH}(G)$ for countable G.

Question 1. How many isomorphism classes of posets $\mathcal{AH}(G)$ are there?

Question 2. Is $\mathcal{AH}(F_2) \cong \mathcal{AH}(F_3)$?

Question 3. Is every finitely presented group AHA?