SURFACE SUBGROUPS

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ABSTRACT. A surface group is the fundamental group of a closed surface of non-positive Euler characteristic. A great deal of recent energy in geometric group theory has focussed on finding surface subgroups in various classes of groups of geometrical interest, especially word-hyperbolic groups. I will survey recent developments, highlights of which include Kahn–Markovic's solution of the Surface Subgroup conjecture for Kleinian groups and Calegari–Walker's discovery of surface subgroups in random groups.

A group Γ *contains a surface subgroup* if there is a closed hyperbolic surface Σ and a monomorphism $\pi_1\Sigma \hookrightarrow \Gamma$.

Question 0 (Gromov). If Γ is a one-ended word hyperbolic group, must Γ contain a (quasiconvex) surface subgroup?

Motivation. Surface Subgroup Conjecture: $\Gamma = \pi_1 M$, M hyperbolic 3-manifold.

$\rightsquigarrow$ Virtual Haken conjecture $\rightsquigarrow$ $vb_1 > 0$ (virtually positive first Betti number)

Gromov, p. 144 "Hyperbolic Groups": "One may suspect that there exist word-hyperbolic groups Γ with arbitrarily large $dim(\partial\Gamma)$ (here, large is ≥ 1) where every proper subgroup is free."

For today, we will assume 'proper' requires ∞-index.

Question 1 (Gromov). If Γ is a non-Fuchsian hyperbolic group, must Γ have a one-ended finitely generated subgroup of infinite index?

Theorem (Thurston–Gromov–Sela–Delzant)**.** *Let H be f.g. one-ended group and let Γ be hyperbolic. Then Γ has finitely many conjugacy classes of subgroups isomorphic to H.*

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1. Kleinian groups

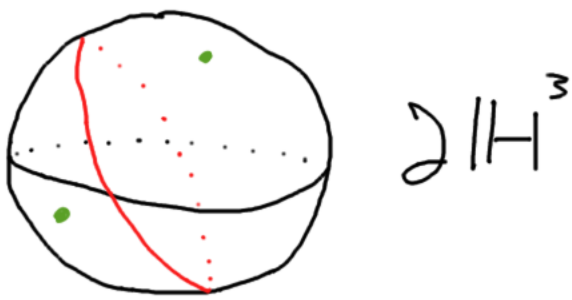
Let $\Gamma = \pi_1 M$, M compact hyperbolic 3-manifold.

- $\partial M \neq \emptyset$ ✓(Cooper–Long–Reid, 1997)
- M arithmetic ✓(Lackenby 2008)
- M closed ✓(Kahn–Markovic 2009)

(Lackenby and Kahn–Markovic inspired by work of Bowen finding free groups in lattices.)

Kahn–Markovic build "many" quasiconvex surface subgroups.

Many means that for any two points on the boundary (green) there is a surface whose limit set separates them (red).



$\rightsquigarrow$ cubulation $\rightsquigarrow$ goodies.

Kahn–Markovic then proved the Ehrenpreis Conjecture. (Extra complications when target is a surface.)

Every homology class in $H_2(M)$ is represented by a geometrically finite surface (Liu–Markovic).

If M is a closed hyperbolic 3-manifold and A is a finite abelian group $\exists$ finite-sheeted cover $M_0 \twoheadrightarrow M$ such that A is a summand of $H_1(M_0)$ (Sun).

Example. $A = \mathbb{Z}/2$.

Use Kahn–Markovic to find non-orientable surface $\Sigma \hookrightarrow M$.

Virtually special $\implies$ $\exists\, M_0 \to M$ finite-sheeted cover such that $\pi_1\Sigma \leq \pi_1 M_0$ and $\exists$ retraction $\rho : \pi_1 M_0 \to \pi_1\Sigma$.

$\implies H_1(\Sigma) \hookrightarrow H_1(M_0)$

Hamenstädt: If Γ is a cocompact lattice in a rank-one Lie group G, then Γ contains a surface subgroup unless $G = Isom(\mathbb{H}^{2n})$ (a case where we don't know).

Question. Do lattices in semisimple Lie groups contain surface subgroups?

Markovic: If Γ is a hyperbolic group with $\partial\Gamma \cong S^2$ and many surface subgroups, then Γ acts properly and cocompactly on $\mathbb{H}^3$.

Except for the condition "many surface subgroups", this is the Cannon conjecture.

2. Random groups

Fix $m \geq 2$ and an alphabet $a_1, \ldots, a_m$. For $l \in \mathbb{N}$ and some $n = n(l) \in \mathbb{N}$, pick n cyclically reduced elements of length l: $r_1, \ldots, r_n \in F_m$, uniformly at random.

Definition. Let $\mathcal{P}$ be a property of groups. We say a *random group has* $\mathcal{P}$ if for $\Gamma = \langle\, a_1, \ldots, a_m \,|\, r_1, \ldots, r_n \,\rangle$,

$$\mathbb{P}\{\Gamma \in \mathcal{P}\} \to 1$$

as $l \to \infty$.

In the "few-relators model", n is fixed.

In the density model, we choose $d \in [0, 1]$ and let $n = \lfloor (2m-1)^{dl} \rfloor$.

Theorem (Gromov)**.** *In the few-relators model, and in the density model with $d < \frac{1}{2}$, a random group is*

- *infinite,*
- *hyperbolic,*
- *of cohomological dimension 2.*

Theorem (Calegari–Walker). *In the few-relator model and density model (with $d < \frac{1}{2}$) a random group contains a surface subgroup.*

$\sim$ break $\sim$

Aside: Sapir's example

$\phi : F_2 \to F_2 : x \mapsto xy, y \mapsto yx$.

$\Gamma = F_2 *_\phi$

Calegari–Walker found $\pi_1 \Sigma \hookrightarrow \Gamma$, $g(\Sigma) = 28$.

Bowditch + Kapovich–Kleiner: if H is one-ended, hyperbolic and 2-dimensional, then after cutting H along finite and virtually cyclic groups, ∂H is in the following list:

- $\checkmark \partial H \cong S^1 \iff H$ cocompact Fuchsian
- $\checkmark \partial H \cong$ Sierpinski carpet, conjecturally $\iff H \sim$ acylindrical 3-manifold group (Cannon conjecture)
- $\checkmark \partial H \cong$ Menger sponge, the "generic" case

Theorem (Calegari–W.). *A random group contains $\pi_1 M$, for M an acylindrical hyperbolic 3-manifold with boundary.*

Conjecture (Calegari–W.). If G is a 2-dimensional hyperbolic group, then a random group contains a quasiconvex subgroup H with $\partial H \cong \partial G$.

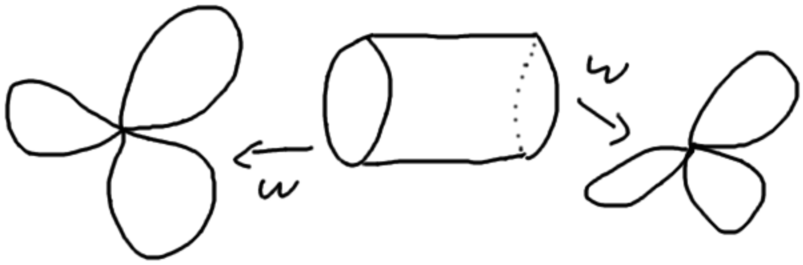
3. Graphs of free groups

For higher-rank edge groups, these "generically" have surface subgroups (Calegari–Walker, Calegari–W.)

Let's focus instead on the case of cyclic edge groups.

For example, $w \in F_n$, $D(w) = F_n *_{\langle w \rangle} F_n$.

$D(w)$ is *hyperbolic* if w is not a proper power, one-ended if w is not contained in a proper free factor.



"strong accessibility"

Theorem (Louder–Touikan)**.** *Let Γ be a one-ended hyperbolic group (without 2-torsion). Then Γ contains a quasiconvex subgroup H of one of two forms:*

- *∂H is connected and has no local cut points*
- *H is π_1 of a graph of free groups with cyclic edge groups*

Calegari: $H_2(\Gamma, \mathbb{Q}) \neq 0 \implies \Gamma$ contains a surface subgroup. [In fact, every integral class in H_2 is represented by a Gromov-norm-minimizing surface. He conjectures this is true for all hyperbolic groups.]

Kim–Oum: $D(w)$ contains a surface subgroup if $n = 2$.

W.: If Γ is not a surface group, then Γ has a one-ended subgroup of infinite index.

Theorem (W., 2016)**.** *One-ended hyperbolic graphs of free groups with cyclic edge groups contain surface groups.*

Corollary. *Hyperbolic limit groups also have surface subgroups.*

$\implies$ a limit group L has $vb_2 > 0$ (virtual second Betti number) unless L is free.

4. Other groups and questions

Question. Do sufficiently complicated mapping class groups contain purely pseudo-Anosov surface subgroups? (convex cocompact)

RAAGs

Question. Which RAAGs A_X contain surface groups?

Question (Kim)**.** If $X = Y \cup Z$, with $Y \cap Z$ complete (RAAG is amalgam along abelian subgroup) and A_X has a surface subgroup, must A_Y or A_Z?

Question (Fisher)**.** Does the automorphism group of a product of bounded valence trees contain a discrete surface subgroup?

Question. If Γ is a non-trivial one-ended group of type F, must Γ contain either a surface or Baumslag–Solitar subgroup.

Audience question: how to show corollary about hyperbolic limit groups?

$\pi_1\Sigma \hookrightarrow L \rightsquigarrow \exists\, L_0 <_{\text{f.i.}} L$ such that $\pi_1\Sigma \hookrightarrow L_0$ has a retraction ρ. The existence of the retraction implies that $H_2(\Sigma) \to H_2(L_0)$ must be injective.