AMENABILITY AND FIXED POINT PROPERTIES

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ABSTRACT. A fundamental dichotomy in the theory of infinite groups is the one between amenable groups and groups with Kazhdan's Property (T). In this talk I shall overview versions of these two opposite properties, connections to actions on non-positively curved spaces and on Banach spaces, to other geometric features of the groups, and to expander graphs. I shall also mention what is known in the setting of random groups and that of important classes of infinite groups (e.g. lattices, mapping class groups, $\mathrm{Out}(F_n)$ etc).

For this topic, we will restrict to finitely generated (f.g.) groups.

Definition (von Neumann)**.** A finitely generated group G is *amenable* if it admits a left-invariant probability measure.

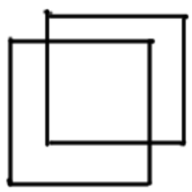
$\iff \forall\, \epsilon, \forall\, F \subseteq G$ finite, $\exists\, \Omega$ finite such that

$$\frac{|F\Omega \,\Delta\, \Omega|}{|\Omega|} \leq \epsilon$$

Example.

(1) G finite

(2) $G = \mathbb{Z}^n$, $\Omega = [-k,k]^n$, k large



(3) G has subexponential growth, $\Omega = B(e,n)$

Properties of amenability.

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(1) Amenability is inherited by subgroups.

(2) If $1 \to N \to G \to Q \to 1$ is a SES, then G is amenable $\iff N, Q$ amenable.

(3) Stable by direct limits

Example.

(4) G solvable

(5) Juschenko–Monod: examples of f.g. simple amenable groups

Question. Can one construct f.p. simple amenable groups?

Theorem. *The following are equivalent:*

(a) G non-amenable

(b) (expansion) ($\exists$ / $\forall$ word metric) $\exists\, C > 0, \alpha > 1$ such that $\forall\, F$ finite, neighborhood $|\mathcal{N}_C(F)| > \alpha|F|$.

(c) (expansion) $\exists\, f : G \to G, d(f, \mathrm{id}) < +\infty$ such that $|f^{-1}(g)| = 2, \forall\, g \in G$. (The bound on $d(f, \mathrm{id})$ means that $\exists\, C$ such that $\forall\, g \in G, d(f(g), g) \leq C$.)

(d) (Gromov) $\exists\, f : G \to G, d(f, \mathrm{id}) < +\infty$ such that $|f^{-1}(g)| \geq 2, \forall\, g \in G$.

(e) G is paradoxical: $G = X_1 \sqcup \cdots \sqcup X_n \sqcup Y_1 \sqcup \cdots \sqcup Y_m$ such that

$$G = g_1 X_1 \sqcup \cdots \sqcup g_n X_n = h_1 Y_1 \sqcup \cdots \sqcup h_m Y_m.$$

Example.

(1) F_2 paradoxical. Let $F_2 = \langle a, b \rangle$.

Let W_a be the set of words beginning with a, and so on.

$F_2 = W_a \sqcup W_{a^{-1}} \sqcup (W_b \setminus \{b^n \mid n \geq 1\}) \sqcup (W_{b^{-1}} \sqcup \{b^n \mid n \geq 0\})$

(2) $F_2 \subseteq G \implies G$ paradoxical.

Conjecture (von Neumann–Day)**.** Is it true that every non-amenable G contains F_2?

Theorem (J. Tits, 1972)**.** *True if G linear, moreover with "amenable" replaced by virtually solvable.*

Example. The Tits alternative is true for

(1) subgroups of $\mathrm{MCG}(S)$ (Ivanov)

(2) subgroups of $\mathrm{Out}(F_n)$ (Bestvina–Feighn–Handel)

(3) $\pi_1(M \text{ compact}, K \leq 0)$ (Ballmann)

In general, the von Neumann–Day conjecture is false.

- Ol'shanskii's monsters (1980)
- Adyan: the free Burnside group
$$B(m,n) = \langle\, x_1, \ldots, x_n \,|\, w^m = 1 \,\rangle$$
which is infinite for $n \geq 2, m$ odd, $m \geq 665$
- Ol'shanskii–Sapir: a f.p. example
- Monod (2013): an example of groups of homeomorphisms of $\mathbb{R}P^1$, piecewise projective
- Lodha–Moore (2014): a f.p. subgroup of Monod's groups, which is 3-generator 9-relator and torsion-free

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Quantitative non-amenability

For all G paradoxical, the Tarski number of G is defined
$$Tar(G) = \min\{n + m \,|\, \forall \text{ paradoxical decompositions}\}.$$

Properties.

(1) $Tar(G) \geq 4$

(2) $H \leq G \implies Tar(G) \leq Tar(H)$

(3) $F_2 \leq G \iff Tar(G) = 4$

$$G = X_1 \cup gX_2 = Y_1 \cup hY_2$$

Conclusion. For $Tar(G) \geq 5 \rightsquigarrow$ classify counterexamples to von Neumann–Day conjecture.

Known Facts (de la Harpe–Ceccherini-Silberstein–Grigorchuk)**.**

(1) G torsion $\implies Tar(G) \geq 6$

(2) Adyan–Sirvanjan $\implies Tar(B(m,n))$ independent of n (they embed into each other)

(3) $Tar(B(n,m)) \leq 14$

Amenability / non-amenability are quasi-isometry invariants (follows from equivalent condition of non-amenability *(b)* above).

Ershov: proved there exists G Golod–Shafarevich group such that $\forall\, m, \exists\, H_m <_{\text{f.i.}} G$, $Tar(H) \geq m$.

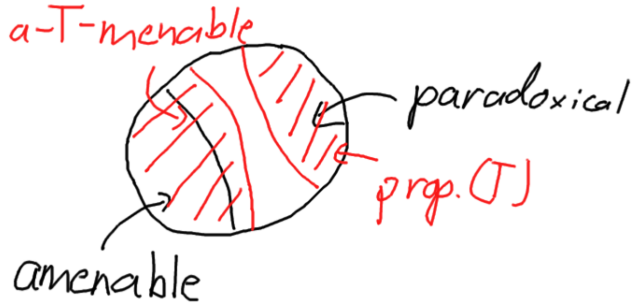
$\implies$ Tarski number is not a quasi-isometry invariant (not even finite index invariant).

Ershov–Golan–Sapir: $\forall\, n, \exists\, G$ with $Tar(G) \in [n, 2n]$. They also gave an example of G with $Tar(G) = 5$. Moreover, they computed $Tar($torsion group of D. Osin$) = 6$.

Question. Is $Tar(G) = 4$ a quasi-isometry invariant. That is, is the property of having a free subgroup a quasi-isometry invariant?

Is $Tar(G)$ small a quasi-isometry invariant (the example of Ershov is for large Tarski number)?

What are the exact values of $Tar(B(n,m)) \in [6, 14]$?



Definition of property (T) and a-T-menability.

G locally compact, second countable. Two properties are defined using actions by isometries (just metric isometries, not necessarily linear!) on a Banach space.

Mazur–Ulam: Every isometry of X Banach is affine

$$v \mapsto u \cdot v + b, \quad u \in U(X)$$

All actions are continuous: $G \curvearrowright X$ by affine isometries, the orbit maps $G \to X : g \mapsto gv$ must be continuous for all v.

Definition. Property (T) $\iff$ every actions on a Hilbert space has a global fixed point (which was originally called Property (FH)).

Definition. a-T-menability (a.k.a. Haagerup property) $\iff$ $\exists$ an action on a Hilbert space which is proper in the sense that $g \to \infty \implies \|gv\| \to +\infty$ (for every basepoint, $\forall\, v$)

Examples of a-T-menable groups

(1) amenable groups

(2) random groups in the Gromov density model for density $d < \frac{1}{6}$

Importance (Higson–Kasparov)**.** If G is a-T-menable then the strongest version of the Baum–Connes conjecture is true ($\implies$ Novikov conjecture).

(3) free groups are a-T-menable but not amenable

Examples of groups with (T).

(1) (lattices in) semi-simple groups with all factors of rank ≥ 2

- $\mathrm{SL}(n, \mathbb{R}), n \geq 3, \quad \mathrm{SL}(n, \mathbb{Z})$
- $\mathrm{SO}(n, m), n, m \geq 2, \quad \mathrm{SO}_{\mathbb{Z}}(n, m)$

(2) For $d > \frac{1}{3}$, random groups have (T)

Relevance of (T).

- Baum-Connes (bad news)
- structural properties (f.g., finite abelianization, $G \neq A *_C B, A*_C$)

- smooth dynamics (Navas, Fisher–Margulis, local rigidity)
- construction of expanders

Connections with NPC spaces and actions on them.

Many NPC spaces have "Hilbert-like" metric: either the distance, or some power of the distance, is a Hilbert norm.

Definition (A conditionally negative definite (CND) kernel)**.** $\psi : X \times X \to [0, +\infty)$, symmetric and $\forall\, n \in N, \forall\, x_1, \dots, x_n \in X, \forall\, \lambda_1, \dots, \lambda_n$ with $\sum \lambda_i = 0$,

$$\sum \lambda_i \lambda_j \psi(x_i, x_j) \leq 0.$$

Example.

(1) Hilbert space H, $\psi(x, y) = \|x - y\|^2$

(2) $L^p, \psi(x, y) = \|x - y\|_p^p, p \in [1, 2]$

(3) (Schoenberg) $\forall\, \psi$ CND, $\exists\, f : X \to H, \psi(x, y) = \|f(x) - f(y)\|_2^2$

Theorem (Delorme–Guichardet, Akemann–Walter)**.**

(1) G has (T) $\iff \forall\, \psi : G \times G \to [0, +\infty)$ CND G-left-invariant is bounded

(2) G is a-T-menable $\iff \exists\, \psi : G \times G \to [0, +\infty)$ CND G-left-invariant proper

Corollary.

(1) If (X, d) is such that d^α is CND, then every action of G on X has bounded orbits, if G has (T).

(2) $\exists\, G \curvearrowright X$ as above, properly discontinuous $\implies$ G is a-T-menable.

Example (of such (X, d))**.**

(1) $\mathbb{H}^n_{\mathbb{R}}, \mathbb{H}^n_{\mathbb{C}}, \quad \mathbb{H}^n_{\mathbb{R}} \hookrightarrow l^1($ measured walls$)$

Question: is there a geometric proof for $\mathbb{H}^n_{\mathbb{C}}$? Current proof due to Faraut and Harzallah.

(2) real trees (T, d_T)

(2) $\implies F_2$ are a-T-menable (through action on Cayley graph). This reproves the following:

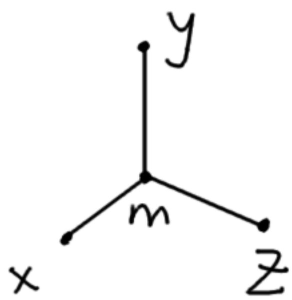
Theorem (Alperin, Watatani)**.** *Property (T) $\implies$ Property (FR) (fixed points for real trees).*

Converse is false: Coxeter groups with $m_{ij} < +\infty$

Converse holds if we enlarge class:

Definition. (X, d) is median if $\forall\, x, y, z, \exists$ a median point m:

$$d(x, m) + d(m, y) = d(x, y)$$
$$d(x, m) + d(m, z) = d(x, z)$$
$$d(y, m) + d(m, z) = d(y, z)$$



Example.

(1) Trees

(2) $(\mathbb{R}^n, \|\cdot\|_2)$

(3) $L^1(X, \mu)$

(4) X simplicial graph

(Vertices, d_X) is median $\iff$ X = 1-skeleton of a CAT(0)-cube complex (Chepoi)

simplicial trees $\to$ real trees

vertices CAT(0) cube complex $\to$ non-discrete version of CCC

Theorem (Chatterji–D.–Haglund)**.**

(1) G has (T) $\iff$ every action on a median space has bounded orbit

(2) *G is a-T-menable $\iff$ there exists a proper action on a median space*

$\sim$ Audience question about actions on asymptotic cone with fixed point $\sim$

Action of $G \curvearrowright Cone_\omega(H)$ without fixed point: if $\exists\, \phi_n : G \to H$ pairwise non-conjugate. In particular, if $\mathrm{Out}(G)$ is infinite, $G \curvearrowright Cone_\omega(G)$ without a fixed point.

$G \curvearrowright G, g \mapsto L_g, L_g(1)_\infty = 1_\infty$

Versions of Property (T) and a-T-menability

(1) Consider actions on Hilbert spaces, affine, uniformly bi-Lipschitz as follows: $\forall\, g \in G$

$$v \mapsto \pi_g \cdot v + b_g$$

$\pi : G \to$Bounded(Hilbert).

$\sup_{g \in G} \|\pi_g\| < +\infty$

Conjecture of Y. Shalom: every hyperbolic group has a proper action that is uniformly bi-Lipschitz.

Bader–Furman–Gelander–Monod: Higher rank lattices have fixed point properties.

(2) Replace Hilbert with L^p, which gives the properties FL^p and a-FL^p-menability.

For $p \in [1, 2]$, $FL^p \iff$ (T) (for $p = 1$, result of Bader–Gelander–Monod), and a-FL^p-menability $\iff$ a-T-menability.

For $p \gg 2$: FL^p is strictly stronger, a-FL^p-menability is strictly weaker.

(Bourdon–Pajot, G. Yu): every hyperbolic group is a-FL^p-menable, $p >$ conformal-dim(∂G).

Higher rank lattices have $FL^p, \forall\, p \geq 1$ (BFGM)

$FL^p \implies$ stronger rigidity results (A. Navas).

Define $\forall\, G$ with (T)

$$\mathcal{N}(G) = \{p \in [1, \infty) \mid G \text{ has } FL^p\}$$
$$p(G) = \sup \mathcal{N}(G).$$

Known: $\mathcal{N}(G)$ is open in $[1, +\infty)$ (Fisher–Margulis), contains $[1, 2]$. The general version can be found in the book of D.–Kapovich.

Open questions.

(1) $\mathcal{N}(G)$ connected?

(2) If $\mathcal{N}(G) \subsetneq [1, +\infty)$, is its complement (or its interior) the set of p for which we have a-FL^p-menability?

(3) (CDH): If $\mathcal{N}(G)$ is bounded, is there a geometric significance of the supremum $p(G)$?

If G hyperbolic, is $p(G)$ a function of conformal-dim(∂G)?

Theorem (D.–Mackay). *In the triangular model for random groups, for density $d > \frac{1}{3}$ asymptotically almost surely*

$$\text{conformal-dim}(\partial G)^{\frac{1}{2} - \epsilon} \leq p(\Gamma) \leq \text{conformal-dim}(\partial G)$$

(the upper bound is by BPY).