HOMOLOGICAL STABILITY, REPRESENTATION STABILITY, AND FI-MODULES

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ABSTRACT. Homological stability is the classical phenomenon that for many natural families of moduli spaces the homology groups stabilize. Often the limit is the homology of another interesting space; for example, the homology of the braid groups converges to the homology of the space of self-maps of the Riemann sphere. Representation stability makes it possible to extend this to situations where classical homological stability simply does not hold, using ideas inspired by asymptotic representation theory. I will give a broad survey of homological stability and a gentle introduction to the tools and results of representation stability, focusing on its applications in topology.

Part I: homological stability

Part II: representation stability

$$Y_n \to Y_{n+1}$$

$H_*(Y_n) \to H_*(Y_n)$ is isomorphism for $* \leq f(n)$. $H_*(\mathrm{SO}(n))$

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	SO(1)	SO(2)	SO(3)	SO(4)	SO(5)	SO(6)	SO(∞)
H_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
H_1		$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
H_2			0	0	0	0	0
H_3			$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/2$
H_4				$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
H_5				0	$\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$
H_6				$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$
H_7					$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$
H_8					$\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$	...
H_9					0	$\mathbb{Z}/2$	...
H_{10}					$\mathbb{Z}$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	...

$$\begin{array}{ccc} \mathrm{SO}(n) & \longrightarrow & \mathrm{SO}(n+1) \\ & & \downarrow \\ & & S^n \end{array}$$

$\mathrm{SO}(n) \to \mathrm{SO}(n+1)$ is $(n-1)$-connected.

$H_*(\mathrm{SO}(n)) \to H_*(\mathrm{SO}(n+1))$ for $* < n-1$.

Often there's some interesting Y such that $H_*(Y_n) = H_*(Y), n \gg *$ (for example, $\mathrm{SO}(\infty)$).

Configuration space $\mathrm{Conf}_n(M) = \{S \subset M : |S| = n\}$.

Example. $\mathrm{Conf}_n(\mathbb{C})$ is a $K(\mathrm{Braid}_n, 1)$.

$H_*(\mathrm{Conf}_n(\mathbb{C})) = H_*(\mathrm{Braid}_n)$

Theorem (Arnold 1969). *$H_*(\mathrm{Conf}_n(\mathbb{C})) \to H_*(\mathrm{Conf}_{n+1}(\mathbb{C}))$ is an isomorphism for $n \geq 2*$.*

Audience question: what is the map? Add a point very far away from the other points.

Theorem (F. Cohen 1973). $\lim_{n\to\infty} H_*(\mathrm{Conf}_n(\mathbb{C})) = H_* \mathrm{Maps}_\infty(\mathbb{R}^2, \mathbb{R}^2)$.

Warning: $\pi_1 \mathrm{Maps}_\infty(\mathbb{R}^2, \mathbb{R}^2) = \mathbb{Z} \neq \pi_1 \mathrm{Conf}_\infty \mathbb{C} = \mathrm{Braid}_\infty$.

Stability for $\mathrm{GL}_n\,\mathbb{Z}$ (originally theorem of Charney)

New approach Bestvina–Church, inspired by Hatcher–Vogtmann.

Define constants c_n as follows: P_n = simplicial complex with vertices pairs (a, b) in $\mathbb{Z}^n$ with $a \cdot 1 = 1$, simplices on $(a_1, b_1), \ldots, (a_k, b_k)$ if $a_i \cdot b_j = 1$ if $i = j$ and 0 otherwise.

c_n = connectivity of P_n

P_n is c_n-connected.

Conjecture (Church–Bestvina)**.** $P_n \simeq \vee S^{n-2}$, so $c_n = n - 3$.

$X_n = \{$ inner products ω on $\mathbb{R}^n\} = \{$ positive definite symmetric $n \times n$ matrices $\simeq \mathbb{R}^{\binom{n+1}{2}}\}$

nonpositively curved metric

$\mathrm{GL}_n\,\mathbb{R} \curvearrowright X_n$

$\mathrm{GL}_n\,\mathbb{Z} \curvearrowright X_n$ with compact $\implies$ finite stabilizers.

$H_*(\mathrm{GL}_n\,\mathbb{Z}; \mathbb{Q}) = H_*(X_n / \mathrm{GL}_n\,\mathbb{Z}; \mathbb{Q})$, $Y_n = X_n / \mathrm{GL}_n\,\mathbb{Z}$

Given ω, say $v \in \mathbb{Z}^n$ is ω-integral if $\omega(v, v) = 1$ and $\omega(v, \mathbb{Z}^n) \subset \mathbb{Z}$.

Define $X_n^k = \{\omega \in X_n \mid \#\text{ of }\omega\text{-integral vectors} > n - k\}$.

$X_n^1 \subset X_n^2 \subset \cdots \subset X_n^n \subset X_n^{n+1} = X_n$.

The filtration starts with lots of ω-integral vectors, ends with one ω-integral vector.

Theorem (C.–Bestvina)**.** *(1) X_n^k is c_k-connected, so $H_*(\mathrm{GL}_n\,\mathbb{Z}; \mathbb{Q}) = H_*(X_n^k / \mathrm{GL}_n\,\mathbb{Z}; \mathbb{Q})$ for $* \le c_k$.*

(2) The quotient space $X_n^k / \mathrm{GL}_n\,\mathbb{Z}$ is independent of n for $n \ge k - 1$.

(If conjecture of BC $\implies$ $H_*(\mathrm{GL}_n\,\mathbb{Z}; \mathbb{Q})$ independent of n for $n > * + 1$.)

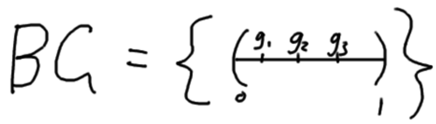
(3) *Integrally* $H_*(\mathrm{GL}_n\,\mathbb{Z}) = H_*^{orb}(X_n^k/\,\mathrm{GL}_n\,\mathbb{Z})$.

As orbifold $X_n^k/\,\mathrm{GL}_n\,\mathbb{Z}$ *not constant by stabilizers for* $G_k \times O_{n-k}(\mathbb{Z})$.

$\sim$ starting again from scratch... $\sim$

Often guess a space Y and prove $\lim_{n\to\infty} H_*(Y_n) = H_*(Y)$ *without* knowing that $H_*(Y_n)$ actually stabilize.

Recall: if G is a discrete group



topology: when points collide, you multiply the labels, and when points hit they boundary, they disappear.

BG is a $K(G,1)$, $\pi_1 = G, \pi_i = 0$.

What if M monoid, like $M = \mathbb{N}$ (all you need in the definition of BG is multiplication).

What is $B\mathbb{N}$? $\pi_1 B\mathbb{N} \neq \mathbb{N}$ because π_1 is a group.

It turns out that $\pi_1 B\mathbb{N}$ is $\mathbb{Z}$, in fact $B\mathbb{N} = K(\mathbb{Z},1)$.

$BM = K(M^*,1)$, where M^* is groupified M (questioned by audience, need sufficiently nice M).

In general, ΩBG is a $K(G,0)$, which is to say $\Omega BG \simeq G$.

For M discrete, $\Omega BM = M^*$.

In general, ΩBM is "groupification of M"

M	ΩBM
$\mathrm{FinSubsets}(\mathbb{R}^2)$	$\mathrm{Maps}_\infty(\mathbb{R}^2,\mathbb{R}^2)$ (F.Cohen)
$\mathrm{FinSubsets}(\mathbb{R}^\infty)$	$\mathrm{Maps}_\infty(\mathbb{R}^\infty,\mathbb{R}^\infty)$ (Barratt–Priddy–Quillen)
$\mathrm{Surf}^\partial(\mathbb{R}^\infty)$ $= \{U \subset \mathbb{R}^\infty\}\{U$ smooth connected 2-manifold w/ 1 component boundary	$\mathrm{Maps}_\infty(\mathbb{R}^\infty,\mathrm{Aff}_2(\mathbb{R}^\infty))$ (Madsen–Weiss)
Metric graphs = { isometry classes of metric graphs w/ basepoint }?	$\mathrm{Maps}_\infty(\mathbb{R}^\infty,\mathrm{Graphs}(\mathbb{R}^\infty))$ (Galatius)
	($\cong \mathrm{Maps}_\infty(\mathbb{R}^\infty,\mathbb{R}^\infty)$)
$\mathrm{Subspaces}(\mathbb{C}^\infty)$	$\mathrm{Maps}_\infty(\mathbb{R}^1,\mathrm{GL}_\infty\mathbb{C})$ (Bott periodicity)

The entries in left column are equivalent to $\bigsqcup_{n\in\mathbb{N}}\mathrm{Conf}_n\,\mathbb{R}^2$, $\bigsqcup_{n\in\mathbb{N}}\mathrm{BS}_n$, $\bigsqcup_{g\in\mathbb{N}}\mathrm{BDiff}(\Sigma_g)$, $\bigsqcup_{n\in\mathbb{N}}\mathrm{BAut}(F_n)$, $\bigsqcup_{n\in\mathbb{N}}\mathrm{BU}(n)$, respectively.

If $M = \sqcup M_n$, then $\mathbb{Z}\times M_\infty \to \Omega BM$ is H_*-iso.

Theorem (Arnold). *$H_*(\mathrm{Conf}_n(\mathbb{C}),\mathbb{Z}) \to H_*(\mathrm{Conf}_{n+1}(\mathbb{C});\mathbb{Z})$ stabilizes.*

Theorem (McDuff, Segal). *For any open manifold M,*

$$H_*(\mathrm{Conf}_n(M);\mathbb{Z}) \to H_*(\mathrm{Conf}_{n+1}(M);\mathbb{Z})$$

stabilizes for $n \gg *$.

Theorem (Church 2012). *For any M,*

$$H_*(\mathrm{Conf}_n(M);\mathbb{Q}) \simeq H_*(\mathrm{Conf}_{n+1}(M);\mathbb{Q})$$

stabilizes for $n > *$.

Why the historical gap?

(1) No maps $\mathrm{Conf}_n(M) \to \mathrm{Conf}_{n+1}(M)$.

(2) False integrally (Artin 1925):

$$H_1(\mathrm{Conf}_n(S^2);\mathbb{Z}) = \mathbb{Z}/(2n-2)\mathbb{Z}$$

$S_n \curvearrowright \mathrm{PConf}_n(M) = M^n - \{p_i = p_j\}$, and $\mathrm{PConf}_n(M) \to \mathrm{Conf}_n(M)$.

$H^*(\mathrm{PConf}_n(M);\mathbb{Q})^{S_n} \to H^*(\mathrm{Conf}_n(M);\mathbb{Q})$

"reduces" to understanding H^* of $\mathrm{PConf}_n(M)$ as an S_n-representation.

Now we have maps $\mathrm{PConf}_n(M) \leftarrow \mathrm{PConf}_{n+1}(M)$ (forget the last point).

$H_*(\mathrm{SL}_n \mathbb{Z}; \mathbb{Z})$ stabilizes,

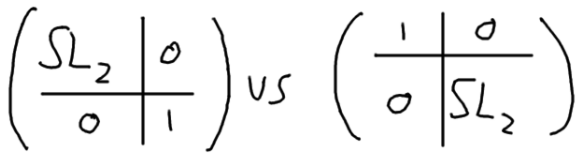
$$\mathrm{SL}_n(\mathbb{Z}, l) = \ker(\mathrm{SL}_n \mathbb{Z} \to \mathrm{SL}_n \mathbb{Z}/l) = \{M \equiv \mathrm{id} \mod l\}$$

but H_* doesn't stabilize, $H_1 \mathrm{SL}(\mathbb{Z}, \mathbb{Z}/l) = sl_n \mathbb{Z}/l = (\mathbb{Z}/l)^{n^2-1}$.

Define $T \subset [n]$

$$\mathrm{SL}_T \mathbb{Z} = \{M \in \mathrm{SL}_n \mathbb{Z} \mid M_{ij} = \delta_{ij} \text{ if } i \text{ or } j \notin T\}.$$

$$\mathrm{SL}_T(\mathbb{Z}, l) = \{M \in \mathrm{SL}_n(\mathbb{Z}, l) \mid M_{ij} = \delta_{ij} \text{ if } i \text{ or } j \notin T\}.$$



$S \subset T, \mathrm{SL}_S \mathbb{Z} \subset \mathrm{SL}_t \mathbb{Z}$.

Theorem (Church–Ellenberg, after Putman CEFN). *"inductive stability"*

$\forall\, n,\ H_i(\mathrm{SL}_n(\mathbb{Z}, l); \mathbb{Z}) = \varinjlim_{T \subset [n], |T| \leq 2^{i+2}} H_i(\mathrm{SL}_T(\mathbb{Z}, l); \mathbb{Z})$ $H_i(\Gamma_n) = \varinjlim_{|T| \leq C, T \subset [n]} H_i(\Gamma_T)$

Definition. $FI =$ category of Finite Sets T and Injections $S \hookrightarrow T$.

FI-module, $V : FI \to \mathbb{Z}$-Mod.

Example. $\mathrm{SL}_\bullet \mathbb{Z} : FI \to$ Groups

$\mathrm{SL}_\bullet(\mathbb{Z}, l) : FI \to$ Groups

$H_i(\mathrm{SL}_\bullet(\mathbb{Z}, l)) : FI \to \mathbb{Z}$-Mod is an FI-module.

$\mathrm{PConf}_\bullet M : FI \to \mathrm{Spaces}^{op}$, $T \mapsto \mathrm{PConf}_T(M) = Inj(T, M)$.

$H^i \mathrm{PConf}_\bullet M : FI \to \mathbb{Z}$-Mod is an FI-module.

An FI-module V is for each n abelian group V_n with S_n-action, $S_n \curvearrowright V_n$, with maps $V_n \to V_N$ $(n < N)$ that play nicely with S_n-action.

As algebraic objects, notion of finitely generated FI-module, similarly finitely presented, projective, injective etc.

Theorem (CEF). *If V an FI-module over $\mathbb{Q}$, then the following are equivalent:*

(1) representation stability for S_n-representations V_n

(2) V is finitely generated.

Theorem (CE). *Any V, the following are equivalent:*

(1) inductive stability : $\exists\, C$ such that $V_n = \lim_{|T| \leq C} V_T$

(2) V is finitely presented

Theorem (CEFN). *FI is Noetherian, so finitely generated $\implies$ finitely presented,* and *both are preserved by e.g. spectral sequences.*

Theorem (Church–Putman). *$\forall\, k, \exists\, C_k, \forall$ genus g such that k-th term of Johnson filtration is generated by elements supported on surfaces / splittings of genus $\leq C_k$.*

Is $\mathrm{SL}_\bullet \mathbb{Z}$ finitely presented as FI-group?

Does there exist C such that $\mathrm{SL}_n \mathbb{Z} = \varinjlim_{|T| \leq C} \mathrm{SL}_T \mathbb{Z}$?