EXTENDING TO LIE ALGEBRAS SOME RESULTS ON SUBDIRECT PRODUCTS OF GROUPS

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ABSTRACT. Since Baumslag and Roseblade discovered that finitely presented subgroups of a product of two free groups must be virtually a product of free groups there has been an intensive research on properties of subdirect products of groups mainly focusing on the homological and homotopical properties, including the celebrated 1-2-3 Theorem by Bridson, Howie, Miller and Short. In this talk we will discuss how to extend some of these results to Lie algebras.

This is a joint work with Dessislava Kochloukova.

1. SUBGROUPS OF PRODUCTS OF FREE GROUPS

Example.

1) Mihailova.

Free $F \overset{\pi}{\twoheadrightarrow} Q$, some group.

Define $P = \{(g_1, g_2) \mid \pi(g_1) = \pi(g_2)\} \leq F \times F$.

This is a pull back of

$$\begin{array}{ccc} & & F \\ & & \big\downarrow{\scriptstyle \pi} \\ F & \xrightarrow{\pi} & Q \end{array}$$

If Q has unsolvable word problem $\implies$ P has unsolvable membership problem.

Miller: Example where P has unsolvable conjugacy problem.

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Minasyan: Q is not residually finite $\implies$ P not conjugacy separable (conjugacy separable means that $\forall\, g, h \in P$ not conjugate, $\exists\, P/H$ finite in which their images are not conjugate).

Grunewald: P not finitely presented.

Definition. In general, with $G_1 \overset{\pi_1}{\twoheadrightarrow} Q \overset{\pi_2}{\twoheadleftarrow} G_2$, we can define

$$P = \{(g_1, g_2) \mid \pi_1(g_1) = \pi_2(g_2)\} \leq G_1 \times G_2$$

which we call the *fibre product*.

$P \cap G_1 \hookrightarrow P \twoheadrightarrow G_2$

$P \cap G_1 \lhd G_1$

P is *subdirect* (that is, it maps epimorphically onto factors), and we definite similarly for more components.

Any $P \leq G_1 \times G_2$ subdirect is a fibre product.

Definition. G is of type $\mathcal{F}_n$ if it has a $K(G, 1)$ with finite n-skeleton.

Example.

2) Bieri–Stallings groups.

Take n finitely generated free groups $F \times \cdots \times F \overset{\mu}{\twoheadrightarrow} \mathbb{Z}$ by sending the standard generators $\mapsto 1$.

Classically, for $n = 2$, $\ker \mu$ is not finitely presented.

For general n, $\ker \mu$ is of type $\mathcal{F}_{n-1}$ but not $\mathcal{F}_n$.

Bestvina–Brady group.

$F \times F \leq \mathrm{SL}(4, \mathbb{Z})$

Inspiration for Bieri–Neumann–Strebel–Renz invariants $\rightsquigarrow$ tell you "something" about properties of subgroups of G over G'.

Baumslag–Roseblade: there are uncountable many subgroups of $F \times F$.

Theorem (Baumslag–Roseblade). *Finitely presented subdirect $P \leq F_1 \times F_2$, direct product of two free groups $\implies$ P is either free or has finite index in $F_1 \times F_2$.*

Theorem (Bridson, Howie, Miller, Short).

(i) *$\mathcal{F}_n$ group $P \leq F_1 \times \cdots \times F_n \implies$ P is virtually a product of (less than n) free groups.*

(ii) *finitely presented subdirect product $P \leq F_1 \times \cdots \times F_n$, with F_i non-abelian free groups and every $P \cap F_i \neq 1 \implies$ P virtually surjects onto pairs $P \to F_i \times F_j$.*

(iii) *if the conditions of (i) and (ii) both hold, then P virtually surjects onto product of m (Kochloukova)*

(iv) *instead of free, one can use limit groups (finitely generated fully residually free groups, as introduced by Sela)*

2. Lie algebras

L. K a field, L a vector space with $[\cdot, \cdot]$, bilinear, $[X, X] = 0$, Jacobi.

Best known example is Lie algebra associated to a Lie group.

Magnus: G group, $\rightsquigarrow$ graded Lie algebra $L(G) = \bigoplus \gamma_i(G)/\gamma_{i+1}(G) \otimes K$, $[\cdot, \cdot] =$ group commutator, γ_i lower central series.

G free $\rightsquigarrow L(G)$ free.

L free Lie algebra.

$A \leq L$ subalgebra $\implies A$ free.

$A \lhd L$ finitely generated $\implies L = A$.

L any Lie algebra $\rightsquigarrow \mathcal{U}(L)$ universal enveloping algebra. Use $\mathcal{U}(L)$ to define "everything" in homological algebra.

Theorem (K,MP). *(i), (ii), (iii) extend to Lie algebras.*

But we need to know how the finiteness properties translate from groups to Lie algebras!

Definition. L is FP_n if $\exists \cdots \to P_n \to \cdots \to P_0 \to K$, with P_i projective and finitely generated.

$\mathcal{F}_n \leftrightarrow L$ is FP_n and L is finitely presented

Proof (of Baumslag–Roseblade Theorem)**.** $P \leq F_1 \times F_2$, which we may assume is a fibre product. Without loss of generality, $F_1 \cap P, F_2 \cap P \neq 1$ also.

$P \cap F_1 \hookrightarrow P \twoheadrightarrow F_2$

$H_2(P)$ has finite rank $\implies$ (with a spectral sequence argument that is drastically simplified since $P \cap F_1$ and F_2 are free) $H_1(F_2, H_1(P \cap F_1))$ is finite rank.

If $P \cap F_1$ ($\lhd F_1$) is finitely generated, then we are done.

(Hall) up to finite index, $F_2 = \langle t \rangle * (\text{something})$.

$H_1(P \cap F_1)$ is a direct summand of $H_1(F_2, H_1(P \cap F_1))$ so it has finite rank.

□

However, this deduction fails for Lie algebras.

3. Converses - 1-2-3 Theorem

If $A \hookrightarrow G_A \overset{\pi_A}{\twoheadrightarrow} Q$ and $B \hookrightarrow G_B \overset{\pi_B}{\twoheadrightarrow} Q$, with $P =$ fibre product.

G_A, G_B finitely generated, Q finitely presented $\implies$ P finitely generated.

Theorem (1-2-3 Theorem, Baumslag, Bridson, Miller, Short, Howie)**.** *A finitely generated, G_A, G_B finitely presented, Q of type $\mathcal{F}_3$ $\implies$ P finitely presented.*

Theorem: this also holds for Lie algebras.

Remark.

- Used to produce a Mihailova $P \leq \Gamma \times \Gamma$ (with Γ hyperbolic), and P finitely presented.
- Used to prove converses of (ii) (G_i finitely presented).

- There is a n-$(n+1)$-$(n+2)$ conjecture (A type $\mathcal{F}_n$, G_A and G_B type $\mathcal{F}_{n+1}$ and Q of type $\mathcal{F}_{n+2} \implies P$ of type $\mathcal{F}_{n+1}$).
- Kuckuck $\rightarrow$ partial progress.

Group case, $\pi_2($ 2-Cayley complex$)$, Q $\mathcal{F}_3$.

Audience question: FP_2 and finitely presented different for Lie algebras? Yes.

Audience question: are there RAAGs for Lie algebras? Not yet.

Audience question: how do you avoid needing Marshall Hall's theorem in the Lie algebra case? $H_1(F_2, V)$ finite rank. $\exists\, t \in F_2$, tV finite dimension $\rightsquigarrow$ $\dim V$ finite.