# Weak forms of amenability for CAT(0) cubical groups

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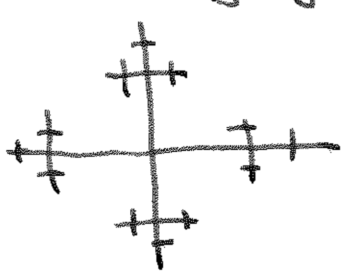
Joint work with J. Brodzki and N. Higson
Few generalities:

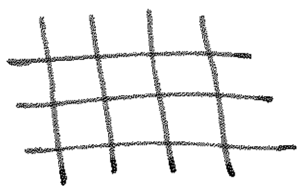
CAT(0) cubical group: a discrete group G, acting on a CAT(0) cubical complex X properly, not necessarily cocompactly. Also assume that X has bounded geometry. (This means that the 1-skeleton is uniformly locally finite.) In particular this implies that X is finite dimensional. (Some of the results that will appear hold in more generallity, but for simplicity the bounded geometry is made a requirement.)

Easy examples: (1) $G = \mathbb{F}_2$. The cube complex here is the tree (1-dimensional cubical complex, the Cayley graph of G).



This may be simple minded but it is a key example for K-amenability, that will be discussed later in the talk.

(2) Another example that is simple minded but relevant later in the talk is $G = \mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$. It acts on $X = \mathbb{R}^2$, cubulated in the usual way.



This is a product of trees, of two 1-dimensional cube complexes. When discussing K-amenability, a key difficulty is how to handle products.

(3) Consider the space $Z$ consisting of the torus with a loop attached at a point.

$Z$

Let $G = \pi_1(Z)$ and $X = \widetilde{Z}$

This is an example of a right angle Artin group, another class of examples that we are interested in.

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<u>Amenability</u> (one of the themes of the conference)

The groups discussed in this talk could be amenable or not (more often <u>not</u> amenable), but $G$ will satisfy weaker properties. They are:

(1) <u>a-T-menability</u>, meaning that $G$ acts properly on a Hilbert space by affine isometries.

A bit of history:

(a) Niblo & Reeves, 1997, proved that CAT(0) cubical groups are a-T-menable. (They did not say that, but they essentially proved it. They showed that the combinatorial distance on the 1-skeleton is negative definite. You need to know also other characterizations of a-T-menability...)

(b) The Theory of Haglund and Paulin, late 1990s. It is possible to construct an action on spaces with walls. This gives a direct construction of the action from the definition of a-T-menability.

(2) Weak amenability: This is a harmonic analysis property, introduced by Haagerup and coauthors in mid-80's (1986). They were studying rank-one Lie groups.

History:

(a) Guentner-Higson, 2010, proved that CAT(0) cubical groups are weakly amenable. At some level, they followed the outline of Haagerup. They constructed uniformly bounded representations of CAT(0) cubical groups.

(b) It is also possible to give a geometric argument. Mizuta (2011?) generalized the original argument of Bożejko & Picardello for trees (1993) to general CAT(0) complexes.

[Bożejko & Picardello proved that free products of amenable groups are weakly amenable.]

Q from audience: For the cited results, is local finiteness needed?

Answer: No. It may not ultimately be needed for K-amenability either...

(3) K-amenability

This involves operator K-theory. There are two different completions of the group ring $\mathbb{C}G$, namely $C^*_{max}G$ and $C^*_{red}G$. The maximal $C^*$-algebra, $C^*_{max}G$, has a universal property. The reduced $C^*$-algebra, $C^*_{red}G$, is "spatial." By the universal property, there always exists a map $C^*_{max}G \longrightarrow C^*_{red}G$. This map is an isomorphism iff $G$ is amenable.

A group is called K-amenable if the induced map at the level of K-theory groups,
$K_*(C^*_{max}G) \longrightarrow K_*(C^*_{red}G)$, is an isomorphism.

(This is actually a consequence of the definition of K-amenability, but serves well as a motivation for what K-amenability represents.)

History:

(a) $G = \mathbb{F}_2$ is K-amenable.

First, One can do this by direct calculation.

- Cuntz, 1981/82, calculated $K_*(C^*_{max}\mathbb{F}_2)$
- Pimsner-Voiculescu, 1982, calculated $K_*(C^*_{min}\mathbb{F}_2)$

You know the generators so you then prove that the above map is an isomorphism.

This is not very illuminating. You do not want, for every group $G$, to know the K-theories of the group $C^*$-algebras ...

(b) Cuntz, 1983, gave the formal definition of K-amenability, for discrete groups.

(c) Julg-Valette, 1984, showed that groups acting on trees are K-amenable. We are going to generalize this argument in this talk.

(d) Higson-Kasparov, 1997 → the electronic announcement

Theorem: The a-T-menable groups are K-amenable.

In the 1997 paper, this result was necessary for their proof of the Baum-Connes conjecture. We note that there is no condition on the action, other than being metrically proper. There is no requirement of bounded geometry, or finite dimensionality, or anything. In the Inventiones paper of Higson and Kasparov (2001), the theorem above is a consequence of Baum-Connes.

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What are our goals?

Understand K-amenability "geometrically," in as concrete way as possible. This means, no use of the machinery behind the proof of the above theorem: the $C^*$-algebra of an infinite-dim Hilbert space, Dirac-dual Dirac construction etc. We would rather generalize the Julg-Valette construction from trees to cube complexes. See arxiv: 1610:05069.

Very short basic structure of the argument

Construct differential complexes, parametrized by $t \in [0, \infty]$, of the form

$$0 \longrightarrow \mathcal{H}_t^0 \longrightarrow \mathcal{H}_t^1 \longrightarrow \cdots \longrightarrow \mathcal{H}_t^n \longrightarrow 0,$$

where $n = \dim X$. These are not algebraic things; they are Hilbert spaces, carrying unitary representations of $G$, and such that

(a) when $t = \infty$, the representations are "spatial". This means that they extend to $C^*_{red}G$. This comes from the properness of the action. (The representations will be sums of quasi-regular representations, with finite stabilizers.)

(b) when $t = 0$, the complex is exactly $G$-equivariant (the differentials are $G$-equivariant maps) and with trivial cohomology ($H^0 = \mathbb{C}$, $H^k = 0$, for $k > 0$). This means that the group acts on cohomology with a trivial action.

(c) When put together, the entire package gives a homotopy of Fredholm complexes (in Kasparov sense). This happens in $KK^G(\mathbb{C}, \mathbb{C})$.

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To summarize: you have a complex that is defined directly from the cube complex, with 'spatial' representations, and the above is supposed to represent a deformation to equivariance and to the unit in $KK^G(\mathbb{C}, \mathbb{C})$.

(a) <u>The complex at $\infty$</u>

Back to the tree for motivation.

$X$ = tree, $X^0$ = vertices, $X^1$ = edges

$G \curvearrowright X$, fix a base point $P_0$.

The differentials look like this:

$$d: \ell^2(X^0) \longrightarrow \ell^2(X^1), \quad d([P]) = \begin{cases} [E] & P \neq P_0 \\ 0 & P = P_0 \end{cases}$$

where $E$ is the edge at $P$ that points toward $P_0$, in the unique path from $P_0$ to $P$


Consider next the Laplacian: $\Delta = (d+d^*)^2 = dd^* + d^*d$

There are two important facts about this complex:

(a) $\Delta = 1 + \boxed{\text{cpct operator}}$

actually a finite rank operator

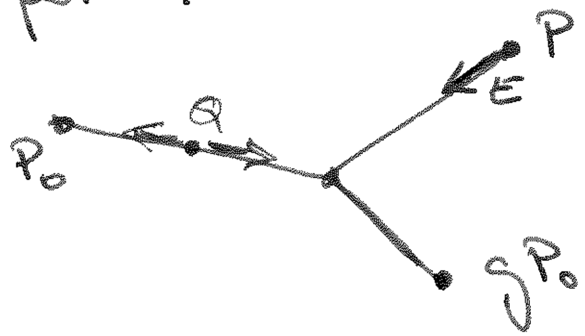
This is just a direct computation: $dd^* = 1$ on $\ell^2(X^1)$ and $d^*d = 1 - \mathrm{proj}_{P_0}$ on $\ell^2(X^0)$.

(b) A weak form of equivariance, required for the complex to give an element in $KK^G(\mathbb{C}, \mathbb{C})$:

$$g(d) - d = \text{compact op.}$$

(In the case of the tree, the above is finite rank.)

Here, $g(d) = g d g^{-1}$. We have sort of two base points $P_0$ and $gP_0$, where $P$ denotes a generic point.




For the vertices on the path from $P_0$ to $gP_0$, the difference $g(d)-d$ is non-zero, but there are only finitely many such points.

These are the two many properties from the paper of Julg–Valette. Let's now see what happens when we want to cover the cube complexes.

(The above is just the first part of the argument. One also needs the complex at $t=0$, and the deformation.)

Let now

$$X = CAT(0) \text{ cube complex}, \quad X^j = j\text{-cubes}$$

$$G \circlearrowright X$$

[Note. The only thing that one needs to know about cube complexes is that they are made of euclidean cubes, glued together on their faces by isometries. Then there is a global metric that satisfies the CAT(0) inequality.]

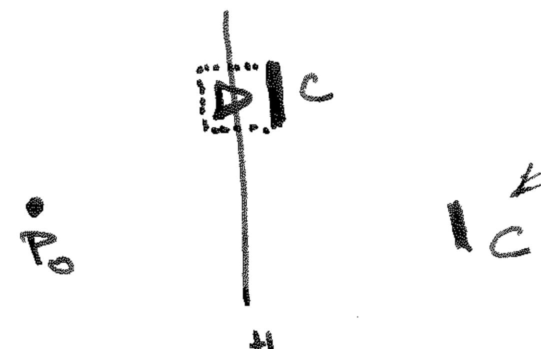
Define the complex:

$$0 \to \ell^2(X^0) \to \ell^2(X^1) \to \dots \to \ell^2(X^n) \to 0.$$

Fix a base vertex $P_0$.

For a hyperplane $H$ and a cube $C$, <u>define</u>

$$H \wedge C = \begin{cases} D, & \text{where } D \text{ is the "extension" of the cube } C \text{ over the hyperplane, in the case when } H \text{ separates } P_0 \text{ and } C, \text{ and } C \text{ is adjacent to } H \\ 0, & \text{otherwise} \end{cases}$$



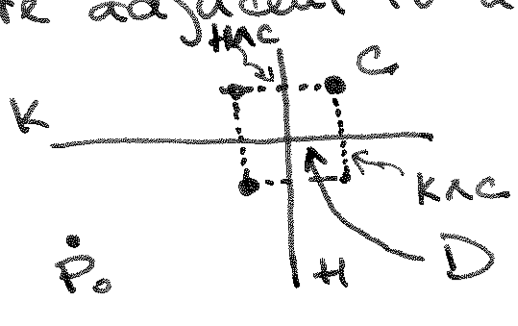

in <u>this</u> case $H \wedge C = 0$

This is a sensible thing to do and maps cubes of one dimension to cubes of one dimension higher. Next define the differential on a basis (which are the cubes):

$$dC = \sum_H H \wedge C$$

Note that there is an upper bound on the number of hyperplanes that separate $P_0$ and $C$, which implies that $d$ extends to a bounded operator.

Is this indeed a differential? We have to check that $d^2 = 0$. One can see that this is not the case. Consider two hyperplanes that are adjacent to a cube $C$.



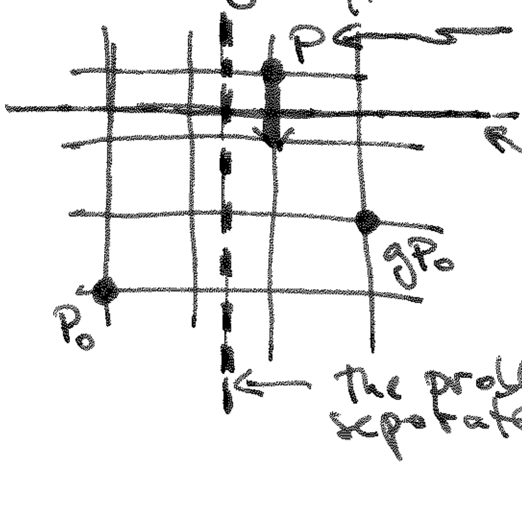

So $K \wedge H \wedge C = D = H \wedge K \wedge C$, but we want cancellation.

We have to introduce a way of orienting the cubes. Every geometric cube will appear twice, with two different orientations, and the square summable functions in the complexes will be antisymmetric ones. In this way $H \wedge K \wedge C$ and $K \wedge H \wedge C$ appear with opposite orientations and the anti-symmetric functions will cancel. We get $d^2 = 0$.

Brief comment on how the orientation of a cube is defined. Each cube is cut by a finite number of hyperplanes. Put an ordering on these hyperplanes and also select a vertex in the cube. This is an orientation. Changing the vertex may change the orientation. If the two vertices are at even distance, they define the same orientation.

Next we have to worry about properties (a) and (b). Property (b), namely $g(d) - d =$ compact operator, is the hard one.

Why is this complicated? This difficulty is already apparent for $\mathbb{Z} \times \mathbb{Z}$.




generic point

this hyperplane will contribute for $P_0$ and $gP_0$ in exactly the same way.

The problem comes from hyperplanes that only separate $P$ from one of $P_0$ and $gP_0$

So a point $P$ live in the picture contributes to a nonzero difference for $g(d) - d$. This is bad in this situation because there are infinitely many such $P$'s. (In the picture, it is the entire vertical strip that seperates $P_0$ and $gP_0$.) So $g(d) - d$ cannot be a compact operator.

At its core, the problem is that cpct. operator $\otimes$ bounded operator is <u>not</u> a compact operator. If one has a product $X \times Y$, the operator $d$ on vertices of $X \times Y$ is of the form (morally):

$$\ell^2(X^0) \otimes \ell^2(Y^0) \xrightarrow{\quad} \ell^2(X^0) \otimes \ell^2(Y^1) \oplus \ell^2(X^1) \otimes \ell^2(Y^0)$$

$$1 \otimes d + d \otimes 1$$

If you are well-studied in Kasparov theory, you know how to fix this, or at least have an idea how to proceed. We will go to an unbounded operator and then renormalize it. Define

$$d_\omega C = \sum \omega(H) \; H \wedge C$$

weights

$$\omega(H) = 1 + d(H, P_0)$$

This is an unbounded operator.

The renormalization is $d' = d\omega \cdot (1 + \Delta\omega)^{-1/2}$.

Properties (a) and (b) now hold for $d'$. The important part is that now $g(d)-d$ is bounded.

## (b) The complex at $t=0$

This is only the combinatorics of hyperplanes. (There are no fixed points. The complex at $t=0$ is supposed to be equivariant on the nose.)

The complex looks like:

$$0 \longrightarrow \ell^2_0(X^0) \to \ell^2_0(X^1) \longrightarrow \cdots \longrightarrow \ell^2_0(X^n) \to 0 ,$$

where

$\ell^2_0(X^2) =$ square summable functions and antisymmetric on the set of oriented symbols.

Definition of an oriented symbol

$$[\underbrace{H_1 \dots H_p}_{\text{set of } p \text{ hyperplanes}} \mid \underbrace{K_1 \dots K_q}_{\text{ordered list of } q \text{ hyperplanes}} \mid R ]$$

$R$: vertex adjacent to H's and K's

The H's and K's are also supposed to intersect. One also makes an equivalence relation on the oriented symbols. Define an operator:

$$d[H|K|R] = \sum_{i=1}^{p} [H_1 \dots \widehat{H_i} \dots | H_i K_1 \dots K_q | R_i] ,$$

where $R_i$ is the vertex across $H_i$ that corresponds to $R$.

<u>Proposition:</u> $d$ is a differential and

$$\Delta [H|K|R] = (p+q) [H|K|R]$$

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This implies that the cohomology is trivial. Recall that $d$ is also equivariant, because no initial point was chosen in its definition.

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(c) <u>The construction of the homotopy.</u>

... we will stop here. Out of time.

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<u>Question from audience.</u> What Theorem does this prove?

<u>Answer:</u> It gives an alternate proof of what is already a Theorem:

"CAT(0) cubical groups are K-amenable."

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<u>Question from audience:</u> Do these constructions go through for finite dimensional but unbounded geometry cube complexes?

<u>A</u>. More likely they do.

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Talk was based on the preprint arXiv: 1610.05069