Irreducible affine isometric actions

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Plan

1. I. Irreducible affine isometric actions : general facts
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3. III. Space of 1-cocycles as Hilbert space
4. IV. Space of harmonic cocyles as von Neumann algebra module

Based in part on joint work with T. Pillon and A. Valette

Actions by affine isometries

Let G be a locally compact group, $\mathcal{H}$ a (real or complex) Hilbert space and

$$\alpha : G \to \mathrm{Isom}(\mathcal{H})$$

an action by affine isometries ; α is given by

- a unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ and
- a 1-cocycle with coefficients in π, that is, a continuous mapping $b : G \to \mathcal{H}$ such that

$$\boxed{b(gh) = b(g) + \pi(g)b(h)} \qquad \text{for all} \quad g, h \in G.$$

Conversely : a unitary representation $(\pi, \mathcal{H})$ and a 1-cocycle b define an affine isometric action $\alpha_{\pi,b}$ through :

$$\boxed{\alpha_{\pi,b}(g)v = \pi(g)v + b(g)} \qquad \text{for all} \quad g \in G, v \in \mathcal{H}.$$

Actions by affine isometries

A coboundary is a cocycle of the form ∂_v for some $v \in \mathcal{H}$, where

$$\boxed{\partial_v(g) = \pi(g)v - v} \qquad \text{for all} \quad g \in G.$$

The space $Z^1(G, \pi)$ of 1-cocycles is a vector space containing $B^1(G, \pi)$ as linear subspace. The 1-*cohomology* of G with coefficients in π is

$$\boxed{H^1(G, \pi) = Z^1(G, \pi)/B^1(G, \pi)}.$$

Irreducible actions by affine isometries

Let $\alpha : G \to \mathrm{I}som(\mathcal{H})$ be an action by affine isometries.

Definition

(Neretin 1997) The action α is *irreducible* if $\mathcal{H}$ has no non-empty, closed and proper $\alpha(G)$-invariant affine subspace.

Examples

Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be an *irreducible* unitary representation of G and $b \in Z^1(G, \pi)$; then $\alpha = \alpha_{\pi,b}$ is irreducible if and only if $b \notin B^1(G, \pi)$.

Remark

- If G has Property (T), then $H^1(G, \pi) = 0$ for every π (Delorme-Guichardet 1972-1977) and so G has no irreducible actions by isometries on Hilbert spaces.
- If G is σ-compact and does not have Property (T), then $H^1(G, \pi) \neq 0$ for some *irreducible* representation π (Shalom 2000) and so has an irreducible actions by isometries on a Hilbert space.

Examples-continued

- Let $b : G \to \mathcal{H}$ be a continuous homomorphism ; then $\alpha = \alpha_{\mathrm{Id},b}$ is irreducible if and only if $\mathrm{Span}(b(G))$ is dense in $\mathcal{H}$.
- Let $G = \mathbf{R}^2$, and $\alpha_1, \alpha_2 : G \to \mathrm{I}som(\mathbf{R})$ defined by the homomorphisms $b_1, b_2 : G \to \mathbf{R}$ given by $b_1(x, y) = x, b_2(x, y) = y$. Then

$$\alpha_1 \oplus \alpha_2 : G \to \mathrm{I}som(\mathbf{R}^2)$$

is irreducible.

A sample application to ℓ^2-Betti numbers

Theorem

(B., Pillon, Valette) *Let Γ be cocompact lattice in a non amenable Lie group G. Then*

$$\boxed{\beta^1_{(2)}(\Gamma) \geq \mathrm{covol}(\Gamma) \sum_{\sigma \in \widehat{G}_d} d_\sigma \cdot \dim_{\mathbf{C}} H^1(G, \sigma)},$$

where $\widehat{G}_d$ is the set of all square-integrable irreducible unitary representations of G.

Characterization of irreducible affine actions

Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of G, $b \in Z^1(G, \pi)$, and $\alpha = \alpha_{\pi,b}$ the corresponding action by isometries. Observe that $\mathrm{Span}(b(G))$ is $\alpha(G)$-invariant.

Proposition

(B., Pillon, Valette) The following properties are equivalent :

- $\alpha = \alpha_{\pi,b}$ is irreducible ;
- $\mathrm{Span}((b + \partial_v)(G))$ is dense in $\mathcal{H}$ for every $v \in \mathcal{H}$.

Quick proof : $\alpha_{\pi,b+\partial_v}(g) = t_v^{-1}\alpha_{\pi,b}(g)t_v$, where t_v is translation by v.

Characterization of irreducible affine actions-continued

Let $\alpha : G \rightarrow \mathrm{I}som(\mathcal{H})$ be an action by affine isometries.

Proposition

"Schur's Lemma"(B., Pillon, Valette) The following properties are equivalent.

- α is irreducible.
- every affine map $\mathcal{H} \rightarrow \mathcal{H}$ which commutes with all $\alpha(g), g \in G$, is a translation (along $\mathcal{H}^{\pi(G)}$).

Admissible probability measures

Let G be a locally compact group, generated by a compact symmetric subset Q. Let $|\cdot|_Q$ be the corresponding word distance on G :

$$|g|_Q = \min\{n \in \mathbf{N} : g \in Q^n\}.$$

Definition

A probability measure μ on G is *admissible* if it has the following properties :

- μ is symmetric ;
- μ is absolutely continuous w.r.t. Haar measure ;
- μ is adapted : its support generates G;
- μ has a second moment : $\int_G |x|_Q^2 d\mu(x) < \infty$.

The space of 1-cocycles as Hilbert space

Let μ an admissible measure and $(\pi, \mathcal{H})$ a unitary representation of G ; then $b \in L^2(G, \mathcal{H}, \mu)$ for every $b \in Z^1(G, \pi)$.

This follows from the fact that $\|b(x)\| \leq C|x|_Q$, for $C = \sup_{q \in Q} \|b(q)\|$.

Proposition

(Guichardet 1972, Ozawa-Erschler 2016) $Z^1(G, \pi)$ is a closed subspace of $L^2(G, \mathcal{H}, \mu)$.

This follows from the fact that the topology of $Z^1(G, \pi)$ (uniform convergence on compact subsets) is given by the norm $\|b\|_Q := \sup_{q \in Q} \|b(q)\|$ and the fact $\|b\|_Q$ is equivalent to the $L^2(G, \mathcal{H}, \mu)$-norm.

Harmonic cocycles

The adjoint of the map $\partial : \mathcal{H} \to Z^1(G,\pi),\ v \mapsto \partial_v$ is $-\frac{1}{2}M_\mu$, with $M_\mu : Z^1(G,\pi) \to \mathcal{H}$ given by

$$\boxed{M_\mu b := \int_G b(x)d\mu(x)}.$$

The orthogonal complement $B^1(G,\pi)^\perp$ in $Z^1(G,\pi)$ can therefore be identified with the space of harmonic cocycles :

Definition

A cocycle $b \in Z^1(G,\pi)$ is μ-*harmonic* if $M_\mu(b) = 0$, that is, $\int_G b(x)d\mu(x) = 0$.

Let $\mathrm{Har}_\mu(G,\pi)$ be the space of μ-harmonic cocyles in $Z^1(G,\pi)$; then

$$\boxed{\mathrm{Har}_\mu(G,\pi) \cong \overline{H}^1(G,\pi) = Z^1(G,\pi)/\overline{B^1(G,\pi)}}.$$

The *commutant* of $\pi(G)$ is

$$\pi(G)' = \{T \in B(\mathcal{H}) : T\pi(g) = \pi(g)T \quad \text{for all} \quad g \in G\};$$

this is a **von Neumann algebra**.

Crucial observation

$\mathrm{Har}_\mu(G,\mu)$ is a module over $\pi(G)'$: if $b \in \mathrm{Har}_\mu(G,\mu)$ and $T \in \pi(G)'$, then $Tb \in \mathrm{Har}_\mu(G,\mu)$, where

$$Tb(g) = T(b(g)) \qquad \text{for all} \quad g \in G.$$

Irreducibility of actions in terms of harmonic cocycles

Recall : a vector v in a module over a ring R is a *separating vector* for R if $Tv = 0$ for $T \in R$ implies $T = 0$.

Proposition

(Adapted from B., Pillon, Valette) Let $b \in \mathrm{Har}_\mu(G, \mu)$. The following properties are equivalent :

- $\alpha = \alpha_{\pi,b}$ is irreducible ;
- b is a separating vector for the $\pi(G)'$-module $\mathrm{Har}_\mu(G, \mu)$.

Irreducibility of actions for harmonic cocycles

Let $b \in \mathrm{Har}_\mu(G, \pi)$ be a μ-harmonic 1-cocycle.

Theorem

The affine action $\alpha_{\pi,b}$ is irreducible if and only if $\mathrm{Span}(b(G))$ is dense.

On the proof : Use the $\pi(G)'$-module structure of $\mathrm{Har}_\mu(G, \pi)$ to show that in fact

$$\overline{\mathrm{Span}\ (b(G))} = \bigcap_{b'} \overline{\mathrm{Span}\ (b'(G))},$$

where b' runs over the 1-cocycles in the cohomology class of b in $\overline{H}^1(G, \pi)$.

Irreducibility of actions : arbitrary cocycles

Let $P_{\mathrm{H}ar} : L^2(G, \mathcal{H}, \mu) \to \mathrm{H}ar_\mu(G, \pi)$ be the orthogonal projection.

Corollary

Let $b \in Z^1(G, \pi)$.
(i) If $\mathrm{Span}(P_{\mathrm{H}ar} b(G))$ is dense in $\mathcal{H}$, then the affine action $\alpha_{\pi,b}$ is irreducible.
(ii) In case $B^1(G, \pi)$ is closed in $Z^1(G, \pi)$, the converse holds.

Remark The converse does not hold in general if $B^1(G, \pi)$ is not closed : there exists an irreducible representation π of F_2 with $\overline{H}^1(F_2, \pi) = 0$ (Martin-Valette 2008) and so $\mathrm{H}ar_\mu(G, \pi) = 0$. Since $H^1(F_2, \pi) \neq 0$, there exists $b \in Z^1(G, \pi)$ which is not a coboundary.

When is $B^1(G,\pi)$ closed ?

Let $(\pi, \mathcal{H})$ be a unitary representation and $\mathcal{H}_0$ the orthogonal complement of the space $\mathcal{H}^{\pi(G)}$ of $\pi(G)$-invariant vectors.

Proposition

(Guichardet 1972) $B^1(G,\pi)$ is closed in $Z^1(G,\pi)$ if and only if π has no almost invariant vectors in $\mathcal{H}^0$.

The case where $\pi(G)'$ is a finite von Neumann algebra

Assume that $\mathcal{M}$ is a finite von Neumann algebra, with faithful normalized trace τ.

Examples

- $\mathcal{M} = M_n(\mathbf{C})$ with τ normalized trace ;
- $\mathcal{M} = \lambda_\Gamma(\Gamma)'' \subset \mathcal{B}(\ell^2(\Gamma))$ the von Neumann algebra generated by the regular representation of a group Γ, with τ given by

$$\tau(T) = \langle T\delta_e, \delta_e \rangle.$$

The von Neumann dimension

Let $L^2(\mathcal{M})$ be the Hilbert space obtained from τ by the GNS construction. We identify $\mathcal{M}$ with the subalgebra of $\mathcal{B}(L^2(\mathcal{M}))$ of operators given by left multiplication with elements from $\mathcal{M}$. The commutant of $\mathcal{M}$ in $\mathcal{B}(L^2(\mathcal{M}))$ is $\mathcal{M}' = J\mathcal{M}J$, where $J : L^2(\mathcal{M}) \to L^2(\mathcal{M}), x \mapsto x^*$. The trace on $\mathcal{M}'$ is defined by $\tau(JxJ) = \tau(x)$ for $x \in \mathcal{M}$.

Every $\mathcal{M}$-module $\mathcal{H}$ is given by a projection

$$P : L^2(\mathcal{M}) \otimes \ell^2(\mathbf{N}) \to \mathcal{H}$$

which commutes with $\mathcal{M}$. Can write $P = (P_{ij})_{i,j}$ as matrix with $P_{ij} \in \mathcal{M}'$. The von Neumann dimension of the $\mathcal{M}$-module $\mathcal{H}$ is

$$\boxed{\dim_{\mathcal{M}} \mathcal{H} = \sum_i \tau(P_{ii})}.$$

The harmonic cocycles as von Neumann module

Let (π, G) be a *factor* representation, that is $\mathcal{M} = \pi(G)'$ is a factor. Assume also that $B^1(G, \pi)$ is closed in $Z^1(G, \pi)$.

Theorem

(i) **Case $\mathcal{M}$ is of type I_∞ or of type II_∞** : *there exists $b \in Z^1(G, \pi)$ such that $\alpha_{\pi,b}$ is irreducible if and only if the commutant of $\mathcal{M}$ in $\mathcal{B}(\mathrm{Har}_\mu(G, \pi))$ is of type I_∞ or II_∞ respectively.*

(ii) **Case $\mathcal{M}$ is of type I_n for $n \in \mathbf{N}$ or of type II_1 :** *there exists $b \in Z^1(G, \pi)$ such that $\alpha_{\pi,b}$ is irreducible if and only if*

$$\dim_{\mathcal{M}} \mathrm{Har}_\mu(G, \pi) \geq 1.$$

(iii) **Case $\mathcal{M}$ is of type III and $\mathrm{Har}_\mu(G, \pi) \neq \{0\}$** : *there always exists $b \in Z^1(G, \pi)$ such that $\alpha_{\pi,b}$ is irreducible.*

A wreath product example

Let $\Gamma = G \wr \mathbf{Z}$ be the wreath product of a finitely generated group G with $\mathbf{Z}$.

Let μ_1 be an admissible probability measure on G, μ_2 the uniform distribution on the generators ± 1 of $\mathbf{Z}$ and $\mu = (\mu_1 + \mu_2)/2$. Let $(\pi, \mathcal{H})$ be a unitary representation of G, viewed as representation of Γ. Then

$$\boxed{\mathrm{H}ar_\mu(\Gamma, \pi) = \mathrm{H}ar_{\mu_1}(G, \pi) \oplus \mathcal{H}}$$

and the action of the von Neumann algebra $\pi(\Gamma)' = \pi(G)'$ on $\mathrm{H}ar_\mu(G, \mu)$ corresponds of $\pi(G)'$ on $\mathrm{H}ar_{\mu_1}(G, \mu_1)$ and on $\mathcal{H}$.

Theorem

Assume that $H^1(G, \pi) = 0$ (this is the case, for instance, when G has Property (T). There exists an irreducible affine action of Γ with linear part π if and only if the representation π is cyclic.