Sublinearly bilipschitz maps, nilpotent and hyperbolic groups

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These maps generalize quasi-isometries.
Let $u : \mathbb{R}_+ \to \mathbb{R}_+$
Assume $u(r) = o(r)$, when $r \to \infty$.

Let X and Y be metric spaces with base points. (The base points are only used to define $|x| = d(x, x_0)$.)
We say that a map $f : X \to Y$ is $\underline{O(u(r))\text{-Lipschitz}}$ if $\exists C$ (Lipschitz constant) and $\exists c$ s.t.
$\forall x, x' \in X$

$$d(f(x), f(x')) \leq C \cdot d(x,x') + \underbrace{c \cdot u(|x| + |x'|)}_{\text{"sublinear"}}$$

Example: if $u \equiv 1$, the $O(1)$-Lipschitz are the large scale Lipschitz maps.

Def: $f, f' : X \to Y$ are $\underline{O(u(r))\text{-close}}$ if

$$d(f(x), f'(x)) \leq c \cdot u(|x|).$$

[Note: This generalizes being of bounded distance.]

Def: The $\underline{O(u(r))\text{-Lipschitz category}}$ has metric spaces as objects and as morphisms the $O(u(r))$-Lipschitz maps, modulo $O(u(r))$-closeness.

Fact (generalizes a classical result about quasi-isometries)

$f: X \to Y$ induces an isomorphism in the $O(u(r))$-Lip category iff

(1) f is $O(u(r))$-Lipschitz

(2) f is $O(u(r))$-expanding:

$$d(f(x), f(x')) \geq C^{-1} \cdot d(x, x') - c \cdot u(|x| + |x'|)$$

(3) (essential surjectivity)

$$d(y, f(x)) \leq c \cdot u(|y|).$$

Def. Similarly, f is $o(u)$-Lipschitz if it is $O(u')$-Lip for some $u' = o(u)$.

"little o"

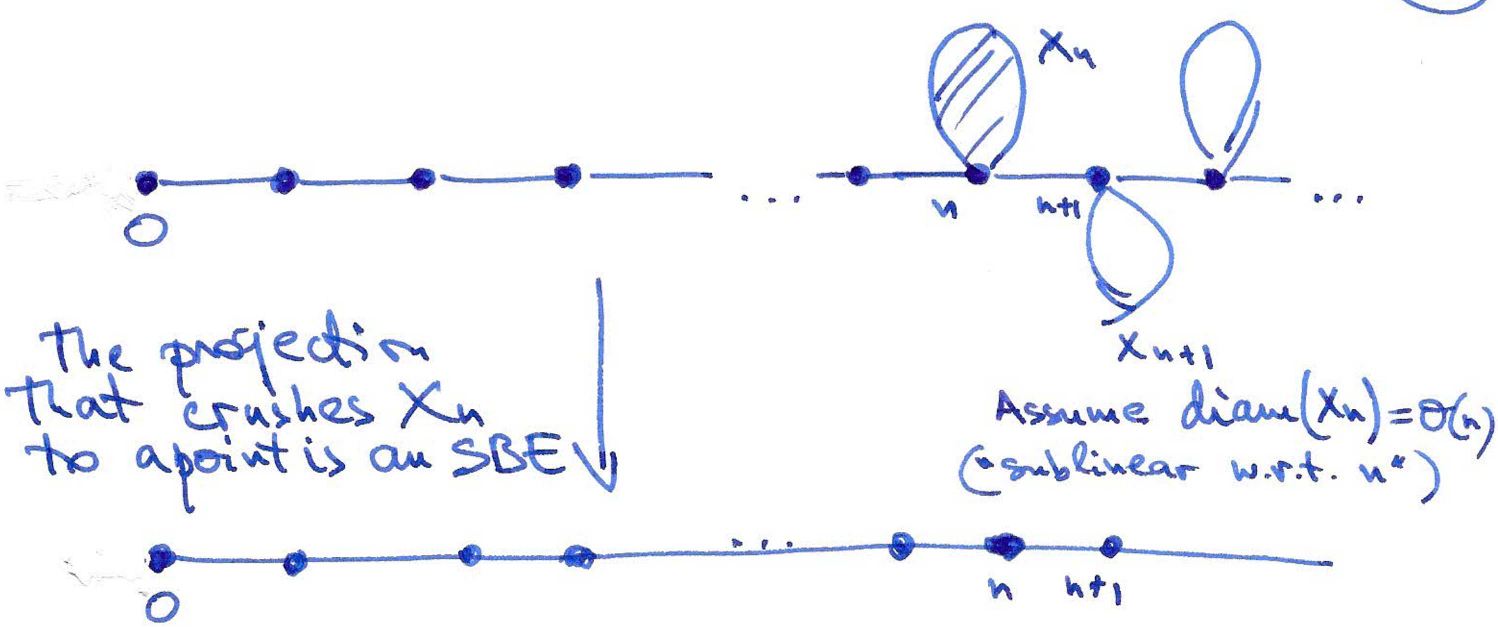
One also defines $o(u)$-category ...

Note: Not clear: maybe $o(r)$

$\boxed{\cancel{O(r)}}$ $o(r)$-Lip maps are those where the sublinear map is not prescribed. $\rightsquigarrow$ in this case the ~~isomorphisms~~ are called sublinearly bilipschitz equivalences (SBE). We denote $u(r)$-SBE when the linear bound u is given.

Note: $O(1)$-SBE are the quasi-isometries.

Example. Consider finite graphs X_n and glue them on an infinite ray. (X_n is glued at n).



The projection that crushes X_n to a point is an SBE

Assume $\mathrm{diam}(X_n) = o(n)$ ("sublinear w.r.t. n")

In particular, for graphs of bounded degree, polynomial growth is <u>not</u> SBE-invariant (it is QI-invariant)

For the same reason, amenability is not SBE-invariant

However, subexponential growth is SBE-invariant.

For <u>groups</u>, polynomial growth is SBE-invariant because it is equivalent to a metric property that is SBE-invariant.

large scale doubling (properness of asymptotic cones)

Next talking about nilpotent and hyperbolic groups ...

Pansu's Theorems in This language (that is, talk about asymptotic cones without using the terminology of asymptotic cones) ....

Pansu 1983. For every simply connected nilpotent Lie group (SCNL) there exists an SBE

$$G \longrightarrow \underbrace{Car(G)}_{\text{associated Carnot graded group}}$$

Pansu 1989. If G_1, G_2 are Carnot and such that $G_1 \xrightarrow[\substack{\text{SBE} \\ \text{equivalence}}]{} G_2$, then G_1 and G_2 are isomorphic as groups.

Consequence. We obtain a full classification of SCNL groups up to SBE:

$$G_1 \overset{SBE}{\simeq} G_2 \quad \text{iff} \quad Car(G_1) \underset{\substack{\text{isomorphic} \\ \text{as Lie algebras}}}{\simeq} Car(G_2)$$

Let $\mathfrak{g}$ be the Lie algebra of G.
Consider the lower central series:

$$\mathfrak{g}^1 = \mathfrak{g}, \quad \mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i].$$

Now make a graded Lie algebra:

$$Car\, \mathfrak{g} = \bigoplus \mathfrak{g}^i / \mathfrak{g}^{i+1}$$, with induced brackets.

[$\mathfrak{g}$ is complicated; $Car\, \mathfrak{g}$ is like a "first order approximation of $\mathfrak{g}$". It is a computable and somehow simple object.

Remark. To compare with the above, we mention a hard conjecture: $G_1 \overset{QI}{\simeq} G_2$ iff $G_1 \simeq G_2$.

Example of SBE invariant : asymptotic cones

Let (X,d) be a metric space.

The asymptotic cone of X, $\mathrm{Cone}_\omega(X)$, is the ultralimit $(X, \frac{1}{n}d)$ with respect to some ultrafilter ω in $\mathbb{N}$.

It is functorial w.r.t large scale Lipschitz maps.
We also get a functoriality

$$O(r)\text{-Lip} \longrightarrow (\text{metric spaces}, \text{Lipschitz maps})$$

where

$$\text{SBE} \longmapsto \text{bilip. homeomorphism}$$

Application: X_1, X_2 symmetric spaces of non-compact type, with no rank 1 factor. Then

X_1 SBE X_2 iff X_1 and X_2 are isometric (up to rescaling metrics on factors)

Hyperbolic groups

For hyperbolic groups, the asymptotic cones do not classify anything (They are all the same).

Theorem: Let X, Y be hyperbolic graphs (in sense of Gromov). If $f: X \to Y$ is $O(r)$-Lip and linearly expanding,

$$|f(x)| \geq \alpha \cdot |x| \quad , \forall x \in X,$$

some constant

then f induces α^{-1}- Hölder map on visual boundaries.

Corollary: SBE induces Hölder homeomorphism of visual boundaries.

Note: SBE is much stronger than having bilipschitz asymptotic cones.

Examples: $F_2 \not\simeq_{SBE} \underbrace{\Gamma_g}_{\text{surface group}}$

$$\mathbb{H}^n_{\mathbb{R}} \not\simeq_{SBE} \mathbb{H}^m_{\mathbb{R}}, \text{ for } n \neq m$$

(because they have nonhomeomorphic boundaries).

In the hyperbolic case, the above are the only results that we can state.

We have a lot of natural questions:

Q1: Are there hyperbolic groups H_1 and H_2 with homeomorphic boundaries, but not Hölder homeomorphic?

Example: (Relevant to Q1) $G_1 = \mathbb{R}^2 \rtimes \mathbb{R}$ and $G_2 = \mathbb{R}^2 \rtimes \mathbb{R}$

action by $\begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}$ and action by $\begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$

$id: G_1 \to G_2$ makes them SBE (even $\log r$-SBE)

The groups are not QI (Xie). Also follows from QI-rigidity of $\mathbb{H}^3$. ($\mathbb{H}^3 \simeq_{QI} G_1$)

Q2: Are there discrete examples?
(Are there H_1, H_2 discrete word hyperbolic that are SBE but not QI?)

Note: G_2 is not QI to any disrete group.

Q3 (Druţu) Are $\mathbb{H}^4_{\mathbb{R}}$ and $\mathbb{H}^2_{\mathbb{C}}$ not SBE?

Remark: if they are SBE, then this gives an answer to Q2.

Back to Nilpotent groups...

A bit more concrete on Pansu's first theorem.
[For every SCNL G, $\exists$ SBE $G \to Car(G)$.]

How can one obtain the map $G \to Car(G)$?

Let $\mathfrak{g}$ be the nilpotent Lie algebra.
Choose a compatible decomposition. From the lower central series

$$\mathfrak{g} = \mathfrak{g}^1 \geq \mathfrak{g}^2 \geq \dots \geq \mathfrak{g}^{c+1} = 0 \quad (\text{c-step nilpotent})$$

~~The~~ a compatible decomposition is something of the form $\mathfrak{g} = \oplus \mathfrak{g}_i$ (linear) s.t. $\mathfrak{g}^i = \bigoplus_{j \geq i} \mathfrak{g}_j$.

Look at the set of all possible decompositions. This is complicated, but encoded in a grading operator

$D: \mathfrak{g} \to \mathfrak{g}$ linear s.t.

$D(\mathfrak{g}^i) = \mathfrak{g}^i$ and $D|_{\mathfrak{g}^i/\mathfrak{g}^{i+1}} = i \cdot$ identity.

Note: W.r.t. the direct sum decomposition,

$$D = \begin{pmatrix} \boxed{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}} & & & 0 \\ & \boxed{\begin{matrix}2 & & \\ & \ddots & \\ & & 2\end{matrix}} & & \\ & & \boxed{\begin{matrix}3 & & \\ & \ddots & \\ & & 3\end{matrix}} & \\ * & & & \ddots \end{pmatrix}.$$

(Form affine subspace of $\mathrm{End}(\mathfrak{g})$.)

This correspondence between compatible decompositions and grading operators goes both ways:

$$\mathfrak{g}_i \longmapsto \text{multiplication by } i \text{ on } \mathfrak{g}_i$$

$$\text{eigenspace } \mathfrak{g}_i \text{ of } \lambda = i \longleftarrow D$$

If D grading operator is fixed (so $(\mathfrak{g}_i)$ is also fixed) we get naturally a linear isomorphism $\mathfrak{g} \to \mathrm{Car}(\mathfrak{g})$.
(More precisely: $[\mathfrak{g}_i, \mathfrak{g}_j] \in \bigoplus_{k \geq i+j} \mathfrak{g}_k$ and the new brakets forget terms of degree $> i+j$.)

$(\mathfrak{g}_i)_i$ is a Lie algebra grading (i.e. $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ $\forall i, j$)

$$\Updownarrow$$

D is a derivation

In this case $\mathfrak{g} \simeq \mathrm{Car}(\mathfrak{g})$, and we say $\mathfrak{g}$ is <u>Carnot</u>.

We need to weaken the condition of D being a derivation. There exists already a notion of higher derivation.

Recall that, by definition, D is a derivation iff $\Delta_2 D = 0$, where $\Delta_2 D(x,y) = D[x,y] - [x, Dy] - [Dx, y]$.

Define:

$$\Delta_n D(x_1, \dots, x_n) = D([x_1, \dots, x_n]) - \sum_{i=1}^{n} [x_1, \dots, Dx_i, \dots, x_n]$$

D is called an n-derivation $\overset{def}{\iff} \Delta_n D = 0$

Fact: derivation $\Rightarrow$ n-derivation, $\forall n$

Def: For $j \geq 3$ and $p_1 + \dots + p_n < j$, a $((p_1, \dots, p_n), j)$-derivation is a D satisfying

$$\Delta_n D(\mathfrak{g}^{p_1}, \dots, \mathfrak{g}^{p_n}) \subseteq \mathfrak{g}^{j+1}.$$

[Note. This is a linear condition on D, so it can be given to a computer, which will say if such D exists or not.]

Def: $\mathfrak{g}$ is λ-derivable (for $\lambda \in \mathbb{R}$) if $\exists$ D grading operator s.t., $\forall$ $p_1, \dots, p_n, j$ satisfying

$$\lambda \leq \frac{\sum p_i}{j} < 1, \quad D \text{ is a } ((p_1, \dots, p_n), j)\text{-derivation}.$$

Def: $e(\mathfrak{g}) \overset{\text{def}}{=} \inf\{\lambda \geq 0 \mid \mathfrak{g} \text{ is } \lambda\text{-derivable}\}$

Facts: $e(\mathfrak{g}) = 0 \iff \mathfrak{g}$ Carnot

$e(\mathfrak{g}) \in \{0, i/j \text{ for } 2 \leq i < j \leq s\}$

s ← nilpotency rank

Exp: $s = 2 \Rightarrow e(\mathfrak{g}) = 0$

$s = 3 \Rightarrow e(\mathfrak{g}) \in \{0, 2/3\}$

$s = 4 \Rightarrow e(\mathfrak{g}) \in \{0, 1/2, 2/3, 3/4\}$

Theorem: There exists $G \to \mathrm{Car}(G)$ an $O(r^{e(\mathfrak{g})})$ - SBE

Questions: (1) Is this optimal?

(2) Does $G \xrightarrow[\log r\text{-SBE}]{} \mathrm{Car}(G) \Rightarrow G$ Carnot?

Even a single example would be interesting...