Box spaces, expanders, and rigidity

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Joint work with Thiebout DELABIE.

Outline

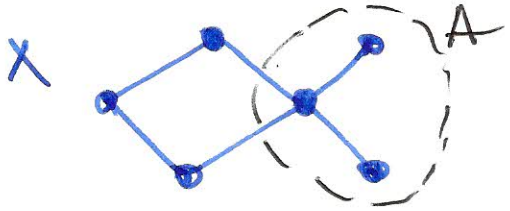
- introduce a geometric object that one can associate to a group, object that encodes some of the geometry of its finite quotients
- examples and connections between the geometric properties of this object and those of the initial group
- more examples of interesting properties

(- a rigidity result concerning these spaces)

Motivation: Expanders

Say you want to build a communication network. Requirements: keep the cost down (not too many wires) and well-connected. We need a notion of connectivity.

Let X be a finite graph, A subset of X.



∂A = The edges that we have to cut to disconnect A

Def: The Cheeger constant of X gives a measure of connectedness:

$$h(X) := \inf_{\substack{A \subseteq X \\ |A| \leq \frac{|X|}{2}}} \frac{|\partial A|}{|A|}$$

<u>Note</u>. If $h(X)$ is large it is hard to disconnect X.

<u>Def</u>: A sequence of finite graphs $\{X_n\}$ is called an <u>expander</u> if

- $|X_n| \longrightarrow \infty$
- the degree of each X_n should be bounded by d, or, simpler for this talk, $\deg(X_n) = d$, $\forall n$.
- (well-connected) $\exists\, \varepsilon > 0$ s.t. $h(X_n) > \varepsilon$, $\forall n$.

<u>Q</u>: Do such graphs exist? Not clear from definition...

<u>Existence?</u> YES

- Pinsker (1973) gave a probabilistic proof that such expanders exist.
- <u>Construction?</u> YES. We need box spaces.

Let G be a finitely generated group that is also <u>residually finite</u>. This means that G has a lot of finite quotients: $\forall\, g \neq e$ in G, $\exists$ finite quotient F s.t. for $\pi : G \to F$, $\pi(g) \neq e$.

(so one can study any finite part of the group in a friendly finite Cayley graph.)

<u>Note</u>: Having a lot of finite quotients can be expressed as having a lot of normal subgroups.

We have a <u>filtration</u> $\{N_i\}$, $N_i \underset{\text{finite index}}{\triangleleft} G$)

$G > N_1 > N_2 > \ldots$, further satisfying $\bigcap_i N_i = \{e\}$.

<u>Def</u>. Let $\{N_i\}$ be such a filtration. The <u>box space</u> of G with respect to $\{N_i\}$ (and a generating set S) is

$$\bigsqcup_i G/N_i \qquad \text{(as a set)}$$

with metric d s.t.

- $d|_{G/N_i}$ = induced Cayley graph metric on the quotient (using S)
- $d(G/N_i, G/N_j) = \operatorname{diam}(G/N_i) + \operatorname{diam}(G/N_j)$ for $i \neq j$

We want to study whatever properties the quotients have in a ~~unified~~ uniform way.

<u>Notation</u>: $\square_{(N_i)} G$.

<u>Note</u>: If Cayley graph can detect properties of G like virtually nilpotent, then the box space should also be useful.

Properties

- it allows us to study quotients "uniformly"
- as n increases, one sees bigger and bigger pieces of the Cayley graph of G:

$\forall R > 0$, $\exists N$ s.t. $\forall n \geq N$

$$B_{G/N_n}(R) \underset{\text{isometric}}{\sim} B_G(R)$$

- "stable" under the change of the generating set G
  in the coarse sense (one gets a coarse equivalence)
- This space captures properties of G

Dictionary (to match the theme of the conference)

G	amenable	property (T)
	$\Updownarrow$	$\Downarrow$
$\square_{(N_i)} G$	property (A) [Yu]	expander [Margoulis]

The first construction of an expander graph

Exp. Take $G = SL_3(\mathbb{Z})$, which has property (T). Make the box space $\coprod_i SL_3(\mathbb{Z}/p^i\mathbb{Z})$. This is an expander.

There are also expanders that are not of this form (not coming from a property (T) group).

Yes: Ramanujan graphs (Lubotzky-Philips-Sarnak)
These are graphs with the optimal spectral gap.

high connectivity $\longleftrightarrow$ first non-trivial eigenvalue, smaller than degree d
λ_1, of the adjacency matrix

(Note. λ_1 is related via some inequalities to the Cheeger constant.)

Ramanujan graphs are graphs that satisfy

$$\lambda_1(X_n) \leq 2\cdot\sqrt{d-1} \quad (\text{"optimal spectral gap"})$$

One can construct such a graph using the free group. (Read about these in the book by Davidoff, Sarnak, and Valette, or in the book of Lubotzky)

$\hookrightarrow$ There are $\{N_i\}$ s.t. $\mathbb{F}_3 / N_i \simeq PSL_2(q^n)$
where q is prime, and satisfying other properties ....

Properties of box spaces

(☆) "balls in the quotients look more and more like the balls in G"

In this last example, because the quotients come from the free group, we get bigger and bigger pieces that look like a tree $\rightsquigarrow$ we get larger and larger girth.

Back to the dictionary:

G	...	Haagerup	...
		$\Uparrow$	
$\square_{(N_i)} G$		coarse embedding into a H.s.	

<u>Def:</u> X <u>coarsely embeds</u> into Y $\left(X \underset{CE}{\hookrightarrow} Y \right)$ if $\exists\, f: X \to Y$ and functions $\rho_{\pm}: \mathbb{R}_+ \to \mathbb{R}_+$ strictly increasing s.t.

$$\rho_-\left(d_X(x,y)\right) \le d_Y\left(f(x), f(y)\right) \le \rho_+\left(d_X(x,y)\right).$$

[<u>Note</u>. The notion is a bit like QI - embedding.]

We will state next an equivalent definition for an expander which will allow us to see that expanders cannot CE into a Hilbert space.

<u>Equivalent definition of expanders:</u> $\exists\, C > 0$, $\forall n$, $\forall f: X_n \to \ell^2$

The Poincaré inequality holds:

$$\frac{1}{|X_n|^2} \sum_{x,y \in X_n} \| f(x) - f(y) \|^2 \le \frac{C}{|X_n|} \cdot \sum_{x \sim y} \| f(x) - f(y) \|^2$$

(x & y are connected by an edge)

From this inequality it follows that expanders do not coarsely embed into Hilbert space (nor ℓ^p).

So something that contains an expander cannot embed into a Hilbert space.

Question: Is ~~containing~~ weakly containing an expander the only obstruction to embedding into ℓ^2?
[The question only makes sense for bounded geometry spaces.]
This was open for many years ...

Answer: No [Arzhantseva-Tessera, 2014]

Construction: They show that there exists a box space of $\mathbb{Z}^2 \rtimes \mathbb{F}_3$ (which has relative (T)) which does not contain (weakly) embedded expanders but does not coarsely embed into ℓ^2.

How does the proof work?

- The fact that the group has relative (T) implies that the box space contains a relative expander.

[relative expander: given a finite group G and $H < G$, we say that G is expanding relative to H if $\exists\, C > 0$ s.t. $\forall$ maps $f : G \to \ell^2$ we have

$$\sum_{\substack{gH \\ \text{(cosets of } H)}} \sum_{x \in gH} \left\| f(x) - \frac{1}{|H|} \sum_{y \in gH} f(y) \right\|^2 \leq C \cdot \sum_{x \sim y} \| f(x) - f(y) \|^2$$
]

- Tessera proved a more general result:

something does not coarsely embed into ℓ^2 $\Leftrightarrow$
$\Leftrightarrow$ contains a relative expander (with respect to a measure)

So: relative (T) $\Rightarrow$ relative expander in box space

$\Rightarrow$ box space $\underset{CE}{\not\hookrightarrow} \ell^2$

How do we show that this does not weakly contain expanders?

Lemma: Given an exact sequence (actually a sequence of exact sequences):

$$1 \to N_n \to G_n \twoheadrightarrow Q_n \to 1$$

if $\coprod_n N_n$ and $\coprod_n Q_n$ coarsely embed into ℓ^2 (in a uniform way), then $\coprod_n G_n$ does not contain an expander.

[Arzhantseva-Tessera] asked in their paper if it is possible to construct a group with such properties (that is, a group which does not embed coarsely in ℓ^2, and does not contain (weakly) embedded expanders).

Note: Bad qualities with respect to embeddings are encoded in the action of $\mathbb{F}^3$ on $\mathbb{Z}^2$, in the previous example).

What about a

Box space of $\mathbb{F}_n$ with these properties?

(joint work with Thibault Delabie)

Consider the Ramanujan sequence $\{N_i\}$ mentioned earlier, and look at

$$\mathbb{F}_3/\mathbb{F}_3 \twoheadleftarrow \mathbb{F}_3/N_1 \twoheadleftarrow \mathbb{F}_3/N_2 \twoheadleftarrow \cdots$$

We take the homology covers of these quotients:

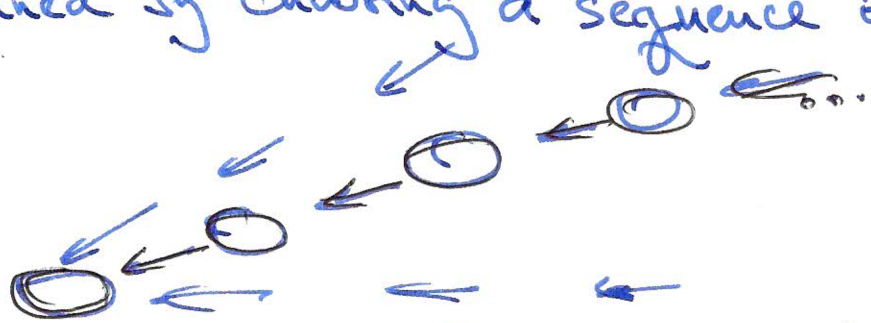
let $\Gamma(N_i) = N_i^2 \cdot [N_i, N_i]$ (subgroup generated by such products)

Note that $N_i / \Gamma(N_i) \cong \oplus \mathbb{Z}_2$. This fact implies that the $\underbrace{\text{covering below}}_{\text{(top diagonal sequence)}}$ does CE in l^2:

$$\begin{tikzcd}
 & & & \mathbb{F}_3/\Gamma(N_3) \arrow[dl] \arrow[d] & \cdots \arrow[l] \\
 & & \mathbb{F}_3/\Gamma(N_2) \arrow[dl] \arrow[d] & \mathbb{F}_3/N_3\cap\Gamma(N_2) \arrow[d] & \\
 & \mathbb{F}_3/\Gamma(N_1) \arrow[dl] \arrow[d] & \mathbb{F}_3/N_2\cap\Gamma(N_1) \arrow[d] & \mathbb{F}_3/N_3\cap\Gamma(N_1) \arrow[d] & \\
\mathbb{F}_3/\mathbb{F}_3 & \mathbb{F}_3/N_1 \arrow[l, two heads] & \mathbb{F}_3/N_2 \arrow[l, two heads] & \mathbb{F}_3/N_3 \arrow[l, two heads] & \cdots \arrow[l, two heads]
\end{tikzcd}$$

The reason is that a space with walls can be constructed. (The first result of this nature is due to Arzhantseva - Guentner - Spakula. Also Khukhro.)

In this triangular diagram the horizontal row contains the expander and the top diagonal CE in l^2. The construction with the desired properties is obtained by choosing a sequence of the type below



It does not CE into l^2 because it stays sufficiently close to the horizontal row, and it does not contain embedded expanders because it is "increasing" to the right ...

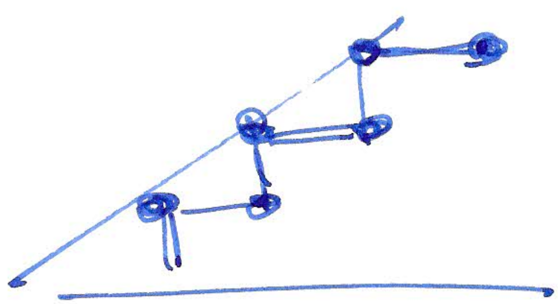
Lemma

Consider the box space $\square_{(M_i)} G$ for $N_i > M_i$ and with $\square_{(N_i)} G \underset{CE}{\hookrightarrow} \ell^2$, Then $\square_{(M_i)} G$ does not contain weakly embedded expanders.

They also show that this carefully chosen sequence satisfies: $\mathbb{F}_3 / N_j \cap \Gamma(N_i)$ is an expander relative to $N_{j+1} \cap \Gamma(N_i) / N_j \cap \Gamma(N_i)$. Consequently it does not coarsely embed in ℓ^2.

Consequence: $\exists$ a "meshed" sequence of subgroups $K_1 > M_1 > K_2 > M_2 > \ldots$ s.t.

$\square_{(M_i)} \mathbb{F}_3 \underset{CE}{\hookrightarrow} \ell^2$, but $\square_{(K_i)} \mathbb{F}_3 \underset{CE}{\not\hookrightarrow} \ell^2$.



Q: How rigid are the box spaces w.r.t. the group? If we have two box spaces of different groups that are coarsely equivalent,

$\square_{(N_i)} G \underset{CE}{\cong} \square_{(M_i)} H$

what can we say about G and H?

[K-Valette] $G \simeq_{\varepsilon i} H$

This was later extended:

[KDas] $G \sim_{UME} H$

(uniformly measure equivalent)

This allows construction of many expanders that are not coarsely equivalent. [see [Hume]]

Theorem (K-Delabie)

[using a coarse version of the fundamental group introduced by Barcelo-Capraro-White]

For finitely presented groups, property (*) implies that the groups are commensurable via a normal subgroup.

Question from the audience: What about the other direction in the dictionary for Haagerup property?

A: It is not true. Think of the free group. They contain a lot of expanders.

The following is true

G	...	Haagerup	...
$\square_{(N_i)} G$		$\Updownarrow$ fibered coarse embeddability	