Quasi-isometric rigidity of fundamental groups of compact 3-manifolds

PETER HAÏSSINSKY

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Joint work with Cyril LECUIRE

Definition. A class $\mathcal{C}$ of finitely generated groups is called quasi-isometrically rigid (QI-rigid) if any finitely generated group which is quasi-isometric to an element of $\mathcal{C}$ contains a finite index subgroup in $\mathcal{C}$.

We present two results:

Theorem 1: The class of finitely generated Kleinian groups is QI-rigid.

Theorem 2: The class of fundamental groups of compact 3-manifolds is QI-rigid.

Note: Theorem 2 is a consequence of the first Theorem and of the work of many other people, as it will be explained later in the lecture.

Some things that were already known ...
The case of surfaces.

Theorem (Gromov, Gabai, Casson-Jungreis)
The class of fundamental groups of compact surfaces is QI-rigid.

The proof of this Theorem depends on the type of surfaces that one considers:

- spherical $\rightsquigarrow$ nothing to prove, π_1 = trivial
- annuli $\rightsquigarrow$ $\pi_1 = \mathbb{Z}$, the result is known

(Even for $\mathbb{Z}^2$ things get complicated. Maybe need Gromov's theorem on polynomial growth ...)

- surfaces of higher genus ...

Point: geometry of the surface dictates the type of proof.

A similar approach is for Theorem 2: the proof is different based on the type of manifold (+ rely on Thurston's geometrization conjecture (Perelman's Theorem)).

Already known for Theorem 2: The case $\chi(M) = 0$ (that is, either M is closed or ∂M = union of tori).

Geometric cases: $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$,

$\mathbb{S}^2 \times \mathbb{R}$, Nil, $\widetilde{SL_2(\mathbb{R})}$, Sol.

not interesting, only get finite groups

not interesting

For $\mathbb{E}^3$ and Nil $\longrightarrow$ work of Gromov and Pansu

$\mathbb{H}^3 \longrightarrow$ Cannon - Cooper, Schwartz

$\widetilde{SL_2(\mathbb{R})} \sim_{qi} \mathbb{H}^2 \times \mathbb{R} \longrightarrow$ Rieffel, Kapovich - Leeb

Sol $\longrightarrow$ [Eskin, Fisher, Whyte]

<u>Non-geometric case</u> (the manifold is made up of several different pieces that are glued together on tori). One can manage to have the pieces with either $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{H}^3$ geometry (Kapovich-Leeb)

The new contribution in Theorem 2:
We allowed boundaries that were more complicated than tori (negatively curved). This explains the need for Theorem 1. Then we checked that the approach of Kapovich and Leeb worked in this new setting.

<u>Def</u>. A <u>Kleinian group</u> is a discrete subgroup K of $PSL_2(\mathbb{C})$.

Two possible actions:

$K \circlearrowright \mathbb{H}^3$ properly discontinuously by isometries

$K \circlearrowright \hat{\mathbb{C}}$ by Möbius transformations

Riemann sphere

not properly discontinuous, but there is a partition

$$\hat{\mathbb{C}} = \underbrace{\Lambda_K}_{\text{"limit set"}} \cup \underbrace{\Omega_K}_{\text{"ordinary set"}}$$

Λ_K: minimal action

Ω_K: properly discontinuous action

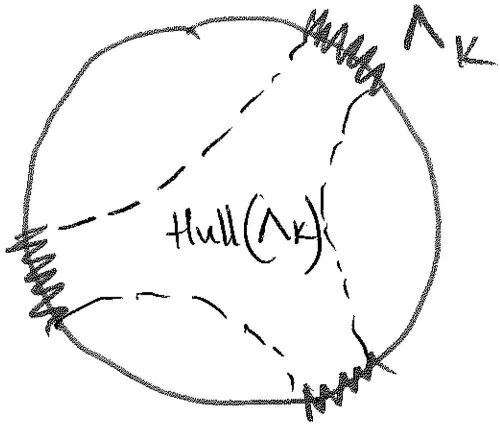
The two actions above can be put somehow together if we identify $\mathbb{H}^3 \simeq \mathbb{B}^3$ (ball model) and $\hat{\mathbb{C}} = \partial \mathbb{B}^3$ (by stereographic projection).

Because we work up to quasi-isometry, we can assume that K is torsion free.
Associate the Kleinian manifold:

$$M_K = \mathbb{H}^3 \cup \Omega_K / K .$$

Then $\pi_1(M_K) = K$.

We want to define two different classes of Kleinian groups that are important.



Because Λ_K is invariant under the group action, its convex hull is also invariant

<u>Definition</u>: K is convex <u>cocompact</u> if $\mathrm{Hull}(\Lambda_K)/K$ is compact.

<u>Definition</u>: We say that K is <u>geometrically finite</u> if $\mathrm{Hull}(\Lambda_K)/K$ has finite volume.

K convex cocompact $\Rightarrow$ K is word hyperbolic and Λ_K is a realization of ∂K.

K geometrically finite $\Rightarrow$ K is hyperbolic relative to π_1(cusps).
In this case the limit set Λ_K is a realization of $\partial_B K$.

Our starting point:

G finitely generated group, K Kleinian group

Assume $G \sim_{qi} K$.

It turns out that we can assume that K is geometrically finite (Thurston) with minimal parabolics (rank 2 cusps).

K is relatively parabolic group, so there exists $\mathbb{P}_K = \{P_1, \dots, P_k\}$, $P_j \simeq \mathbb{Z}^2$ (isom. to tori)

By a Theorem of Druţu-Sapir, it follow that G is also relatively hyperbolic $(G, \mathbb{P}_G)$ such that $\forall P \in \mathbb{P}_G$, P is virtually $\mathbb{Z}^2$.

Even more, from the work of Groff, G acts on Λ_K by uniform quasi-Möbius maps.

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quasi-Möbius maps have good controls on cross-ratios, in the following sense (Väisälä):

Consider a metric cross-ratio:

$$[x_1 : x_2 : x_3 : x_4] = \frac{|x_1 - x_2| \cdot |x_3 - x_4|}{|x_1 - x_3| \cdot |x_2 - x_4|}$$

Also consider an increasing homeomorphism $\eta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$\forall x_1, x_2, x_3, x_4 \in \Lambda_K$, $\forall g \in G$,

$$[g(x_1) : g(x_2) : g(x_3) : g(x_4)] \leq \eta\left([x_1 : x_2 : x_3 : x_4]\right).$$

The following result holds (a key point in the proof of Cannon-Cooper):

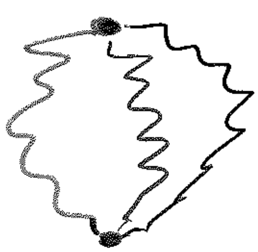
<u>Theorem (Sullivan):</u> A uniform group of quasi-Möbius maps on $\hat{\mathbb{C}}$ is conjugate to a group of Möbius transformations.

So in the case when $\Lambda_K = \hat{\mathbb{C}}$ we are done.

<u>Idea:</u> Take the action of G on Λ_K, extend it to the entire sphere, then apply Sullivan's theorem. This is a little too naïve.

In many cases Λ_K is disconnected.

Assume for the moment that Λ_K is connected (so the group is one-ended). We only have an action on a compact set. There is no obvious way to extend it to the entire sphere...



When M_K is acylindrical, up to some quasi-conformal change of K, we can assume that

$$\underbrace{\Lambda_K = \hat{\mathbb{C}} \setminus \left(\bigcup \text{round disk}\right)}_{\text{complement of round disks}}$$

We also know that Λ_K has measure zero (because of geometrically finite).

Theorem (Bonk-Kleiner-Merenkov)
In our setting, any quasi-Möbius map of Λ_K is the restriction of a Möbius mapping.

As a corollary of this, we know that G is a Kleinian group.

The above works when M_K is acylindrical. The general strategy will be different.

Assume K is one-ended and $\Lambda_K \neq S^1$.
(If $\Lambda_K = S^1$, then we have surface groups.)

(i) Consider a JSJ decomposition of G. In other words, G is the fundamental group of a graph of groups:

$$G = \pi_1\left(\Gamma, \{G_v\}, \{G_e\}\right)$$

$$G_e \hookrightarrow \boxed{G_{t(e)}}$$

neighboring vertices

(ii) Show that $\forall\, v \in \Gamma^{(0)}$, G_v is virtually Kleinian.

(iii) Associate the Kleinian manifold M_v and build a manifold M by gluing the pieces M_v following the graph structure. We will have $\pi_1(M) \simeq G$.

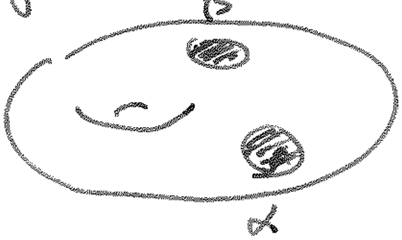
(iv) Lastly, apply Thurston's uniformization Theorem.

Actually, in (iii) $\pi_1(M) = G'$, with $[G:G'] < \infty$.
... (see the video...). This passing to a finite index subgroup is not an artifact of the proof, as the following

example shows.

Example due to Kapovich-Kleiner

Start with a 2-dimensional torus; make two holes; glue one hole to the other by turning around twice:

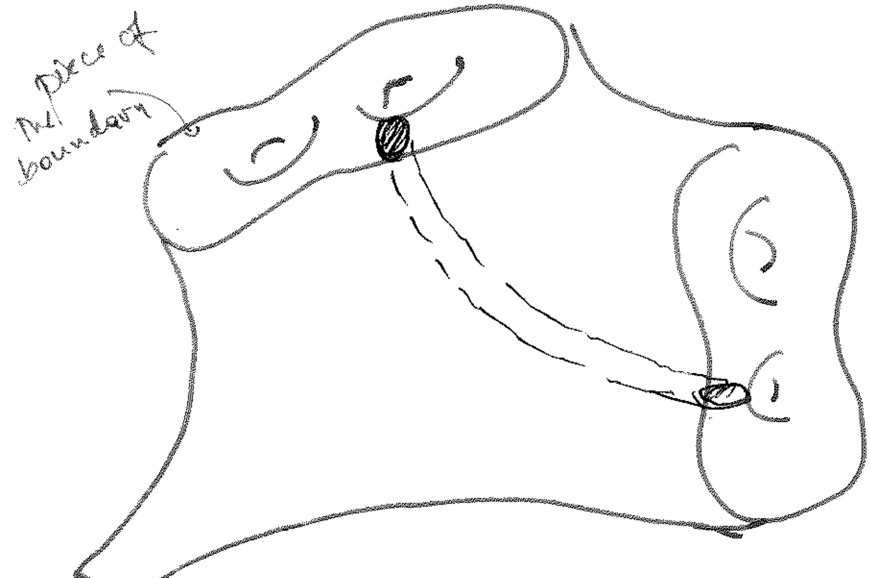


$\beta \sim \alpha^2$

This is a word hyperbolic group, an HNN extension of a free group (so torsion free), which cannot be a 3-manifold group.

With a finite index subgroup, it turns out that one can desingularize this situation and get a 3-manifold.

An explanation of the JSJ decomposition



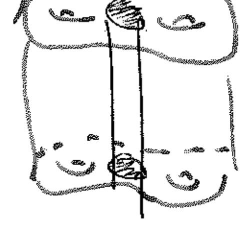
Idea: cut the manifold along a family of essential annuli

$\exists$ finite collection of essential annuli s.t.

$M_K \setminus (\bigcup A_j)$

is comprised of pieces of the following type:

- solid torus (unusual)
- surface I-bundles (product of a surface from boundary with I)
- $\mathbb{T}^2 \times [0,1]$
- acylindrical paved manifold



In each case we have the fundamental group of the piece and this is what defines the graph structure for the group.

(Note: the four types give the following groups:
$\mathbb{Z}$, $\mathbb{F}_K$ $(K \geq 2)$, $\mathbb{Z}^2$, Möbius group, respectively.)

When looking on the universal cover and how the pieces are glued together, we get an action of K the dual tree of the graph of groups: $K \curvearrowright T$.

<u>Theorem</u> (Bowditch, Groff)
for word hyperbolic, relatively hyp.

Both K & G act on the tree T.
Moreover the quasi-isometry between G and K specifies quasi-isometries

$$\varphi_v : \left(K_v, \{K_e, e \in E(v)\}\right) \longrightarrow \left(G_v, \{G_e\}\right).$$

What can we say about the vertice groups G_v?

To prove (ii), the most interesting piece comes from the acylindrical pieces. We have a collection of annuli on the boundary of the manifold.

Pinch the curves and send them to ∞ to create parabolic points (Thurston, Ohshika).

Pinch the core of the annuli of K_v $\rightsquigarrow$ obtain a new Kleinian group K'_v isomorphic to K_v, such that

$$\Lambda_{K_v'} = \Lambda_{K_v} \Big/ \{\Lambda_e\}$$

and M_v' become acylindrical in the previously discussed sense.

Now that we have all these new pieces we have to glue them together to form a new manifold M, such that $\pi_1(M) = G'$, $[G : G'] < \infty$.

There are several problems that arrive from this ...

$$(G_v, \{G_e\}) \rightsquigarrow (G_v', \{G_e'\})$$

edge groups are virtually cyclic

cyclic

may not be primitive ...

A theory of | $\underline{\text{Wise}}$ Haglund Agol | explains how to get separability results for quasi-convex subgroups (this is a theory of CAT(0) cube complexes).

Question from audience: What happens in higher dimensions?

Everything breaks down ...

He is using a lot of techniques from 3-manifolds ...