Sofic mean length

Hanfeng LI
12/7/2016
2:00PM

Joint work with Bingbing Liang

R : a unital ring

Def (Northcott & Reufel '65)

A length function for left R-modules assigns a value $L(M) \in [0, +\infty]$ for each ${}_R M$ satisfying:

(1) $L(0) = 0$

(2) Additivity: for any short exact sequence of R-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

Then $L(M_2) = L(M_1) + L(M_3)$.

(3) Upper continuity

$$L(M) = \sup_{A \subseteq M} L(A)$$

finitely generated R-submodules

Note: The most important property is additivity. Upper-continuity can be replaced by a weaker condition or removed.

Examples: (1) R field.

$$L(M) = \begin{cases} \dim_R(M) & \text{if } \dim_R(M) < \infty \\ +\infty & \text{otherwise} \end{cases}$$

(2) R integral domain. Turn the R-modules into vector spaces (example (1)) by looking at the fraction field k of R.

In this case $L(M) = \begin{cases} \dim_k(k \otimes_R M) & \text{if } \dim_k(k\otimes_R M)<\infty \\ +\infty & \text{otherwise} \end{cases}$

These first two examples are special.

(3) For arbitrary R, consider an increasing sequence of R-sub-modules

$$M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_n = M$$

and define

$$L(M) = \sup_{\substack{\text{such}\\ \text{subsequences}}} n$$

This is a length function and is characterized by being the unique length function s.t. $L(A)=1$ for every simple R-module A.

In fact, there exists a length function L_α for each ordinal α, and the one in example (3) above is just the first one.

From now on, let G be a discrete (countable) group. We want to consider some group actions.

Let $M = \bigoplus_{s\in G} R$.

Then G acts on M by shift (on the indexing set).

If G is infinite and $L(R)=1$, then $L(M)=+\infty$, because there are submodules made out of n copies of R, for any n. On the other hand, M is generated by a <u>single copy</u> of R under <u>the action</u> of G. ~~It~~ should be the case that "length of M under $G \curvearrowright M$" should be $L(R)$, so more useful. How can we make this precise?

Use the group ring: RG = finitely supported functions $f : G \to R$

with operations:

$$\sum_s f_s \underset{\in R}{\cdot} \underset{\in G}{s} + \sum_s g_s s = \sum (f_s + g_s) s$$

convolution product

$$\left(\sum f_s s\right) \cdot \left(\sum g_t t\right) = \sum_{s,t} \underset{\in R}{(f_s g_t)} \underset{\in G}{(s \cdot t)}$$

Examples: (1) For $G = \mathbb{Z}$, $RG \simeq R[x^{\pm}]$ Laurent polynomials

Indeed: $\sum_{n \in \mathbb{Z}} f_n \cdot n \Longleftrightarrow \sum_{n \in \mathbb{Z}} f_n \cdot x^n$

(2) Similarly, $R\mathbb{Z}^d \simeq R[x_1^{\pm}, \dots, x_d^{\pm}]$ Laurent polynomials

---

We can make the above question precise:

Q: Given a length function L on R-modules, can we define a length function $\tilde{L}$ on RG-modules, s.t. $\forall_R M$ $\tilde{L}(RG \otimes_R M) = L(M)$?

Remarks: $G \curvearrowright_R M \Longleftrightarrow {}_{RG}M$

Also, $RG \otimes_R M$, as R-module, is $\bigoplus_{s \in G} M$, so when $M = R$ we recover the motivating example above.

Answer: essentially "YES" for amenable groups.

if G is amenable, take a left Følner sequence $\{F_n\}$.

<u>Def</u> (Liang-L 2013, Virili 2014)
For ${}_{RG}M$ define

$$\tilde{L}(M) := \sup_{\substack{A \subseteq M \\ \text{fin. generated} \\ R\text{-submodule}}} \left(\lim_{n\to\infty} \frac{L\left(\sum_{s\in F_n} s^{-1}A\right)}{|F_n|} \right)$$

<u>Notes:</u> (1) The limit above does exist.
(2) For this to work we need to require A to have finite length.

<u>Def:</u> ${}_RM$ is called <u>locally L-finite</u> if $L(A) < +\infty$ for every finitely generated R-module $A \subseteq M$.

<u>Theorem:</u> $\tilde{L}$ defined above is a length function on locally L-finite RG-modules.

[The locally L-finiteness is not that restrictive. If $L(R) < \infty$, then every ${}_RM$ is locally L-finite.]

Sometime we do need $L(R)$ to be $+\infty$ ...

<u>Answer:</u> "No" for free groups.
Consider $\mathbb{F}_2 = \langle a, b \rangle$. In this case

$$\begin{aligned} R\mathbb{F}_2 &\supseteq R\mathbb{F}_2(a-1) + R\mathbb{F}_2(b-1) \\ &= R\mathbb{F}_2(a-1) \oplus R\mathbb{F}_2(b-1) \\ &\simeq R\mathbb{F}_2 \oplus R\mathbb{F}_2 \quad \text{as } R\mathbb{F}_2\text{-module} \end{aligned}$$

So if we had a length function, we would get

$$\tilde{L}(R\mathbb{F}_2) \geq \tilde{L}(R\mathbb{F}_2 \oplus R\mathbb{F}_2) = 2\tilde{L}(R\mathbb{F}_2).$$

As we want $\tilde{L}(R\mathbb{F}_2) = L(R)$, this is impossible if $0 < L(R) < +\infty$.

Def. (Gromov '99) G is sofic if G can be approximated by finite groups.

That is, by subgroups of permutation groups of finite sets

Formally this means that $\exists\ \Sigma = \{\sigma_i : G \longrightarrow \mathrm{Sym}(d_i)\}_{i=1}^{\infty}$

permutation group of $\{1, 2, \dots, d_i\}$

satisfying:

(1) $\forall s, t \in G$

$$\frac{\left|\{v \in \{1, \dots, d_i\} \mid \sigma_{i,s}\,\sigma_{i,t}(v) = \sigma_{i,st}(v)\}\right|}{d_i} \xrightarrow{i\to\infty} 1$$

[in other words, σ_i is almost a group homomorphism as $i \to \infty$.]

(2) [σ_i become more and more injective]

$\forall s, t \in G$

$$\frac{\left|\{v \in \{1, \dots, d_i\} \mid \sigma_{i,s}(v) \neq \sigma_{i,t}(v)\}\right|}{d_i} \xrightarrow{i\to\infty} 1$$

(3) $d_i \longrightarrow \infty$. (This condition is automatically satisfied if G is infinite. In this case (2) $\Rightarrow$ (3).)

Examples: (1) Amenable groups are sofic.
Consider a Følner sequence $\{F_n\}$. Take
$\sigma_n : G \longrightarrow Sym(F_n)$, $\sigma_{n,s}(t) = st$, $\forall\, t \in F_n \cap s^{-1}F_n$.

[The permutations are left multiplication.]

(2) Residually finite groups are sofic.
Indeed, take $\{1, 2, \dots, d_i\}$ = finite quotient groups.

Let G be sofic and fix a sofic approximation
$\Sigma = \{\sigma_i : G \to Sym(d_i)\}_{i=1}^{\infty}$.
Let L be a length function on R-modules. We need to look at relative invariants.
Let $M_1 \subseteq M_2$ be RG-modules.
Take $A \subseteq M_1$ f.g. R-module
$B \subseteq M_2$ f.g. R-module
$F \subseteq G$ finite subset
$\sigma : G \longrightarrow Sym(d)$ [This will be eventually one of the σ_i's.]

Define

$\mathcal{M}(B, F, \sigma) :=$ sub-R-module of M_2^d generated by
$\delta_v b - \delta_{sv}\, sb$, for $v \in \{1, \dots, d\}$, $b \in B$, $s \in F$

$\mathcal{M}(A, B, F, \sigma) :=$ image of A^d in $M_2^d \longrightarrow M_2^d / \mathcal{M}(B, F, \sigma)$.

[This is still a fin. generated R-module.]

(Liang-Li)

Define the mean length of M_1 relative to M_2 by

$$\widetilde{L}(M_1|M_2) := \sup_{A} \inf_{B} \inf_{F} \lim_{i\to\omega} \frac{L(\mathcal{M}(A,B,F,\sigma_i))}{d_i}$$

ω is a free ultrafilter on $\mathbb{N}$

Note. For the absolute case (only one module), $\widetilde{L}(M) = \widetilde{L}(M,M)$.

Theorem. For locally L-finite RG-modules $M_1 \subseteq M_2$,

$$\widetilde{L}(M_2) = \widetilde{L}(M_1|M_2) + \widetilde{L}(M_2/M_1).$$

[A sort of additivity that takes into account the fact that things do not quite work for free groups.]

The amenable group case does not depend on M_2:

Theorem: if G is amenable, for locally L-finite RG-modules $M_1 \subseteq M_2$, $\widetilde{L}(M_1|M_2) = \widetilde{L}(M_1)$, as previously defined.

Applications

Def: R is called directly finite (or von Neumann finite) if $\forall\, a,b \in R$, $ab = 1 \implies ba = 1$.

Examples: (1) $M_n(\mathbb{C})$ is directly finite.

(2) $B(\ell^2(\mathbb{N}))$ is not. Take the unilateral shift

[Note: $M_n(\mathbb{C}) = B(\mathbb{C}^n)$.].

Kaplansky's direct finiteness conjecture ('69)
$\forall$ field k, $\forall G$, kG is directly finite.

Kaplansky showed that this is true if the field has characteristic 0.

Ara, O'Meara, Perera '02 $\rightarrow$ Yes, for R skew-field and G residually amenable

Elek, Szabo '04 $\rightarrow$ Yes, R skew-field, G sofic

Ceccherini-Silberstein, Coornaert '07 $\rightarrow$ Yes, if R is artian, G sofic

Virilli '14 $\rightarrow$ Yes, if R is left Noetherian, G amenable

Theorem. RG is directly finite if R is left Noetherian and G is sofic.

Another application is about dynamics.
Gromov '99, Linstrauss - Weiss '99

mean dimension for amenable $G \curvearrowright$ compact metrizable X

a dynamical analogue for covering dimension of X

Instead of giving the definition, let's look at some examples.

Examples: (1) mean dim$(G \curvearrowright M^G) = \dim(M)$,
where M is an arbitrary cpct manifold, and action is by shift.

(2) mean dim$(G \curvearrowright X) > 0 \Rightarrow$ entropy$(G \curvearrowright X) = +\infty$.

what about mean dimension for sofic $G \curvearrowright X$?

von Neumann - Lück rank in L^2-inv (related to ℓ^2-Betti numbers)

$$\mathbb{Z}G \subseteq \mathbb{Q}G \subseteq \mathcal{L}G = \overline{\mathbb{C}G \xrightarrow[\text{left reg.}]{} B(\ell^2(G))}^{\text{weak-op norm}}$$

left group vN algebra

For a fin. gen. projective $\mathcal{L}G$-module M, we can write
$M \simeq (\mathcal{L}G)^n P$, for some $n \in \mathbb{N}$, $P \in M_n(\mathcal{L}G)$, $P^2 = P$.

The von Neumann dimension is defined as:

$$\dim_{vN}(M) = \mathrm{tr}(P) := \sum_{j=1}^{n} \mathrm{tr}(P_{jj})$$

[For $a \in \mathcal{L}G$, $\mathrm{tr}(a) = \langle a\delta_e, \delta_e \rangle$.]

Lück extended this to arbitrary modules: for any $\mathcal{L}G$-module M,

$$\dim_{vN}(M) = \sup_{\substack{A \subseteq M \\ \text{fin. gen. projective } \mathcal{L}G\text{-module}}} \dim_{vN}(A).$$

For a $\mathbb{Z}G$-module M, $\operatorname{rank}(M) = \dim_{vN}(\mathcal{L}G \otimes_{\mathbb{Z}G} M)$.

Theorem. Let G be sofic. For any countable $_{\mathbb{Z}G}M$ ($\Leftrightarrow G \curvearrowright M$ ab. group $\Leftrightarrow G \curvearrowright \hat{M}$ compact ab. group)

then
$$\operatorname{rank}(M) = \operatorname{mean\,dim}(G \curvearrowright \hat{M}).$$

The two sides are related by showing that both equal $\tilde{L}(M)$ for $L(A) = \dim(\mathbb{Q} \otimes_{\mathbb{Z}} A)$.

Comment following a question from audience:
The constructions done for amenable groups do not depend on the Følner sequence. For sofic groups, There is a dependence on the sofic approximation.

$\boxed{\text{? there may be ?}}$