Finite Embeddability of groups and its applications to geometry and topology

GUOLIANG YU
12/7/2016
3:30PM

We will discuss some connections between topology and group Theory, where the bridge between them is analysis.

M = compact oriented topological manifold

We are interested in deciding whether M is rigid or not, in the sense that we want to know if there exists another manifold N that is homotopy equivalent to M but not homeomorphic to M. And if such manifolds exist, how big is their collection?

Introduce the structure group

$$S(M) = \left\{ (X, f) \;\middle|\; \begin{array}{l} X = \text{cpct. oriented manifold} \\ f: X \to M \text{ orientation preserving} \\ \text{homotopy equivalence} \end{array} \right\} \Big/ \sim$$

The collection of pairs (X, f) is modded out by an appropriate equivalence relation and the result is an abelian group.

__Goal__: measure the size of this group, in terms of data coming from $G = \pi_1 M$ (maybe also from the topology of the manifold).

This is joint work with Shmuel Weinberger and Zhizhang Xie. (Some papers were written already on this subject, by these three authors.)

Sample results we are after:

__Conjecture 1__. Look at the free part of $S(M)$. Then

$$\text{rank of } S(M) \geq N_{\text{finite}}(G) = \#\left\{ d \in \mathbb{N} \;\middle|\; \begin{array}{l} d = \text{order}(g), \\ \text{for some } g \in G \end{array} \right\}$$

[__Note__: This is some basic information that measures the amount of torsion in the group.]

[__Note__: They believe that this result should hold in general.]

__Conjecture 2__. Let $n = \dim(M)$. Then

$$\text{rank of } S(M) \geq \text{rank of } KO_n^G(\underline{EG}, \tilde{M}),$$

where $\underline{EG} = \lim\limits_{d \to \infty} P_d(G)$.

Rips complex

__Note__: KO_n^G is relative KO-homology

__Note__: Conjecture 2 is more precise than Conjecture 1. Conjecture 1 is a consequence of Conjecture 2.

Conjecture 2 implies that $S(M)$ is infinitely generated if $KO_n^G(\underline{EG}, \tilde{M})$ is infinitely generated.

<u>Question:</u> Construct natural groups G s.t. $KO_n^G(\underline{EG})$ is infinitely generated.

<u>Note</u>. In topology is hard to prove that the structure group is infinitely generated. This would mean that M is highly nonrigid, in the sense that there are many manifolds that are homotopy equivalent to M but not homeomorphic to M.

What condition(s) need to be imposed on $\pi_1 M$ for the conjectures to become true?

<u>Theorem 1</u>. If G is coarsely embeddable into a Hilbert space then Conjecture 2 holds

[Coarse embeddability means that the large scale geometry of the group is compatible with the Euclidean geometry, in the infinite dimensional sense.]

We do have now a lot of examples of groups that are embeddable into a Hilbert space.

<u>Theorem 2</u>. If G is <u>finitely embeddable</u> into Hilbert space then conjecture 1 holds.

Here we use:

<u>Definition</u>: G is said to be <u>finitely embeddable</u> if for any finite subset $F \subseteq G$, there exists a map $\phi : F \longrightarrow H$, where H is another group which is coarsely embeddable into a Hilbert space such that:

(0) $\phi(e) = e$;

(1) if $g, h, gh \in F$, then $\phi(gh) = \phi(g) \cdot \phi(h)$;

[main condition] (2) order is preserved for torsion elements:

$$\text{order}(\phi(g)) = \text{order}(g), \quad \forall g \in F.$$

<u>Examples of finitely embeddable groups</u>:

(1) residually finite groups

(2) $Out(\mathbb{F}_n)$

(3) Gromov monster groups.

[Point of the definition: one only cares about large multiplication tables sitting in the group and not really about the structure of the big group.

<u>Note</u>. There exist examples of groups that are <u>NOT</u> finitely embeddable. The construction is based on facts from logic, not groups that "come from nature."

Next we get to the main tool, bridging group theory and topology & analysis.

<u>Note</u>: The structure group only makes sense for topological manifolds. For smooth manifolds it is surprising that the definition does not work. Nevertheless, to give the main idea behind the construction without being too technical, we will exemplify with smooth manifolds.

<u>Higher rho invariants</u>

[S. Weinberger, John Lott
in this context, by N. Higson & J. Roe (smooth context)
Zenobi (in topological context)]

The idea is quite natural.
M smooth manifold, X smooth, $f: X \to M$ smooth

The first invariant to look at is (de Rham) cohomology. We want to consider the above as a system. So we construct a complex that computes relative cohomology:

$$\Omega^*(X) \oplus \Omega^{*+1}(M), \text{ acted by } d_f = \begin{pmatrix} d & f^* \\ 0 & d \end{pmatrix}.$$

Consider $D_f = d_f + d_f^*$ (we actually need the signature ~~operator~~ operator, which is part of it, but let's not worry about it now).

When f is a homotopy equivalence, the relative cohomology is trivial. But we want to construct something secondary, like passing from characteristic classes to Chern-Simon classes. We are going to do this using operator theory.

f is a homotopy equivalence $\Rightarrow D_f$ is invertible, so it's Fredholm index is zero. We want to construct a more subtle invariant. First normalize the unbounded self-adjoint D_f by setting $F = \dfrac{D_f}{|D_f|}$. F is still invertible and $F^2 = I$.

<u>Crucial idea</u>. The differentiation operation is local, but $\dfrac{D_f}{|D_f|}$ becomes more global. How can we make F <u>more local</u> while still keeping it invertible? This does not always happen, and here is where the invariant is coming from.

We want to do this at the level of universal cover:

$$\Omega^*(\tilde{X}) \oplus \Omega^{*+1}(\tilde{M}).$$

For each $n \in \mathbb{N}$, $\exists$ a G-invariant open cover $\{U_i^{(n)}\}$ of $\tilde{M}$, and $\{\varphi_i^{(n)}\}$ a smooth partition of unity subordinate to $\{U_i^{(n)}\}$, s.t.

$$\operatorname{diam}(U_i^{(n)}) < \frac{1}{n}.$$

Construct a local version of F:

$$F_n = \sum_i (\varphi_i^{(n)})^{1/2} \, F \, (\varphi_i^{(n)})^{1/2}$$

F_n is essentially F (they are htpy. equivalent). This process does not change the "primary" information about F. F_n is a local operator.

We put together all the F_n's in a family of operators. Define, for $n \le t \le n+1$, a linear homotopy:

$$F_t = (t-(n+1)) F_n + (t-n) \cdot F_{n+1}$$

Also let $F_0 = F$.

So now we have a family $\{F_t\}_{t \in [0,\infty)}$. As $t \to \infty$, the operator becomes more and more local. Can we still keep the invertibility of the operator?

In general, when making the operator local we loose invertibility. How can we measure the loss of invertibility? Here is the trick: consider

$$e^{2\pi i \left(\frac{F_t+1}{2}\right)}$$

This is defined as a formal power series, so it makes sense. It is idempotent and invertible. How far is it from trivial?

We want to put this invariant somewhere...

We assumed in the above that M is <u>odd dimensional</u>. (if $\dim M =$ even things are a bit more technical)

$$\left[e^{2\pi i\left(\frac{F_t+1}{2}\right)}\right] \in K_1\left(C^*_{L_0}(\tilde{M})^G\right).$$

Note: This is a secondary invariant defined when the primary invariant (the Fredholm index) is zero.

[John Roe] Consider an operator $T : L^2(\tilde{M}) \to L^2(\tilde{M})$.

(Something like $(T\xi)(x) = \int_{\tilde{M}} \underbrace{k(x,y)}_{\substack{\text{"big matrix"} \\ \text{"continuous matrix"}}} \cdot \xi(y)\,dy$

s.t. (0) k continuous and bounded

supported near diagonal (1) $\mathrm{supp}(k) \subseteq \{(x,y) \in \tilde{M}\times\tilde{M} \mid d(x,y) \le r \text{ for some } r\}$

[The smallest such r is called the propagation of T.]

G-inv (2) $k(gx,gy) = k(x,y)$, $\forall g \in G$.

Def: $C^*(\tilde{M})^G = \overline{\{T : L^2(\tilde{M}) \circlearrowleft \text{ satisfying (0), (1), and (2)}\}}^{\|\ \|}$

Fact: $C^*(\tilde{M})^G = C^*_r(G) \otimes K$ ← C^*-algebra of compact operators.

<u>Def</u>: $C^*_L(\tilde{M})^G = \left\{ \begin{array}{l} f : [0,\infty) \to C^*(\tilde{M})^G \\ \quad - \text{uniformly bounded} \\ \quad - \text{uniformly continuous} \\ \quad - \text{propagation}(f(t)) \xrightarrow{t\to\infty} 0 \end{array} \right\}$

Consider the evaluation map e and form an exact sequence:

$$0 \to C^*_{L,0}(\tilde{M})^G \longrightarrow C^*_L(\tilde{M})^G \xrightarrow[e]{\text{evaluation}} C^*(\tilde{M})^G \to 0$$

$$f \longmapsto f(0)$$

$$C^*_{L,0}(\tilde{M})^G = \{ f \in C^*_L(\tilde{M})^G \mid f(0) = 0 \}$$

Pass to K-Theory:

this is the Baum-Connes map

$$e_* : K_*(C^*_L(\tilde{M})^G) \longrightarrow K_*(C^*(\tilde{M})^G)$$

very computable K-homology

hard to compute!

$C^*_{L,0}(\tilde{M})^G$ is an obstruction to e_* being an isomorphism. It mixes the localness with the globalness.

$$\rho(D_f) := \left[e^{2\pi i \left(\frac{F_t + 1}{2} \right)} \right] \in K_1(C^*_{L,0}(\tilde{M})^G)$$ is the obstruction for the inverse of F (or D_f) being local.

We constructed a map $\rho : S(M) \longrightarrow K_*(C^*_{L,0}(\tilde{M})^G)$

A technical result (which was an open question for a number of years):

<u>Theorem:</u> ρ is a homomorphism.

(see a paper on arXiv, 2016)

How can we prove Theorem 1 or 2? The last effort goes into getting information out of $K_*(C^*_{L,0}(\tilde{M})^G)$.

<u>Theorem:</u> if G is coarsely embeddable, then ρ_{Nov} is surjective.

Here ρ_{Nov} is what makes the following diagram commutative:

$$\begin{tikzcd} S(M) \arrow[r, "\rho"] \arrow[dr, dashed, "\rho_{Nov}"'] & K_*(C^*_{L,0}(\tilde{M})^G) \arrow[d] \\ & KO^G_*(\underline{EG}, \tilde{M}) \end{tikzcd}$$

Novikov conjecture is used here, which requires coarse embedding

computable

<u>Question from the audience:</u> is ρ surjective?

<u>A</u>: This is a hard question.
$S(M)$ is mysterious; $K_*(C^*_{L,0}(\tilde{M})^G)$ is still mysterious but we can get some information out of it. If we understood it, we would solve more than Baum-Connes conjecture.