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Equicontinuous actions of semisimple Lie groups

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Joint work with Tsachik Gelander

The ideas that he will talk about are classical and their contributions are minimal, but they are very interesting.

① Mautner Lemma (Segal, von Neumann)
(see some historical remarks in the introduction of their paper)

$G \to \mathrm{Iso}(X,d)$ (G acts cont on a metric space)

Assume that $\exists$ a sequence (a_n), and $u \in G$ s.t.
$a_n^{-1} u a_n \longrightarrow e$

if $x \in X$, $a_n x \longrightarrow x \implies ux = x$.

Proof: We have to show that $d(ux, x) = 0$

But $d(ua_n x, a_n x) \xrightarrow{?} d(ux, x)$.

$\quad \|$ inv. metric

$d(a_n^{-1} u a_n x, x) \longrightarrow d(ex, x) = 0$.

② Theorem Let $G = SL_2(\mathbb{R})$

$G \to \mathrm{Iso}(X,d)$ and there are no fixed points ($X^G = \emptyset$). Then the action is proper.

[Def of proper
$\forall C, C' \subseteq_{cpct} X \quad \{g \mid gC \cap C' \neq \emptyset\} \subseteq_{cpct} G$]

<u>Proof</u>: Show that if the action is not proper then $\exists$ a G-fixed point in X.

Not proper $\Rightarrow$ $\exists \underset{\in G}{g_n} \longrightarrow \infty$ s.t. $g_n C \cap C' \neq \emptyset$.

So we can find $x_n \in C$ and $g_n x_n \in C'$.
Using compactness, assume $x_n \longrightarrow x'$
$g_n x_n \longrightarrow y'$

$G = K A^+ K$, $K = SO(2)$, $A = \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$

So $g_n = k_n a_n k_n'$, $a_n = \begin{pmatrix} \alpha_n & \\ & \alpha_n^{-1} \end{pmatrix}$ and $\alpha_n \to \infty$.

$k_n \to k$, $k_n' \to k'$

Set $y = k^{-1} y'$ and $x = k' x'$; we get $a_n x \to y$.

We are almost in the setting of Lemma, but with two points. Do some zig-zagging...
We can find a sequence $m_n \longrightarrow \infty$ s.t. $\underbrace{\alpha_{m_n} / \alpha_n}_{\beta_n} \longrightarrow \infty$

$$\begin{tikzcd} x \bullet \arrow[r, bend left, "a_{n_k}"] & \bullet\, y \arrow[l, bend left, "a_{m_n}"] \end{tikzcd}$$

Let $b_n = a_{m_n} \cdot a_n^{-1}$. Then $a_{m_n} X \to y$, $a_n^{-1} y \to X$

Here they use the metric (as well in the Mautner Lemma)

And $b_n x = a_{m_n} \cdot a_n^{-1} x \longrightarrow x$.

$$b_n = \begin{pmatrix} \beta_n & \\ & \beta_n^{-1} \end{pmatrix}$$

For $\forall t$, $b_n^{-1} \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} b_n = \begin{pmatrix} 1 & t/\beta_n^2 \\ 0 & 1 \end{pmatrix} \longrightarrow e$

By M lemma, we get $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} x = x$

Also $b_n^{-1} x \to x$ $\quad b_n^{-1} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} b_n \to e \Rightarrow \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} x = x \quad \forall t$

As $SL_2(\mathbb{R}) = \left\langle \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right\rangle \Rightarrow x \in X^G$.

■

Note: The Theorem holds for any simple group over any local field.

What matters in Theorem 2 is not the metric structure on X, but the uniform structure on it.

(3) Equicontinuous actions

Def: A uniform structure on X is a symmetric filter S of relations on X, each contains the diagonal of $X \times X$ s.t.

$\forall \mathcal{U} \in S$, $\exists \mathcal{U}' \in S$ s.t. $\mathcal{U}' \cdot \mathcal{U}' \subset \mathcal{U}$, where

relation

[relation of being at ε distance]

$$\mathcal{U}_1 \cdot \mathcal{U}_2 = \left\{ (u_1,u_2) \;\middle|\; \exists u_3,\ \begin{matrix}(u_1,u_3)\in\mathcal{U}_1 \\ (u_3,u_2)\in\mathcal{U}_2\end{matrix} \right\}$$

Think of the relations as a sort of semi group that acts on X.

<u>Def:</u> $G \curvearrowright X$ preserving S is <u>equicontinuous</u> if S is generated by G- invariant relations.

We now note Thm 2 listing equicont actions:

<u>Theorem:</u> G simple Lie group act equicontinuously on (X,S). if $X^G=\varnothing$ Then The action is proper

In particular, orbits are closed and stabilizers are compact.

Same proof as Theorem 2 works ...

<u>Corollary:</u> Every continuous homomorphism from simple Lie group to any Hff. top. group $G\to H$ has a closed image.

[An old Thm of van Est '51, Omori '66.]

<u>Proof:</u> H acts on itself preserving a uniform structure ~ "the left uniform structure". This is The structure generated by

$$\mathcal{U}_V = \{(h_1,h_2) \mid h_2^{-1}h_1 \in V\}, \quad V \text{ some identity nghbd of } H$$

it is invariant under left action of H.
Then G acts on H, by the isomorphism, and preserves the uniform structure

There are no fixed points $\xrightarrow{\text{Theorem in (3)}}$

$\Rightarrow$ orbits are closed

$\Rightarrow$ orbits of identity = image of G.

Making it even a bit more formal.

G acts on a Hs. $\Longrightarrow$ matrix coefficients go to 0

$\Longrightarrow$ The only pt. on the boundary of an orbit is 0 w.r.t weak top

Dyn. System by action

Weak top does not come from a uniform structure
The norm top. comes from a unif str., but it is compatible with the other one

Think of unif structure with the order as a sort of semigroup which acts on sets

This is compatibility....

(4) Compatible topologies

Given a uniform space (X, S) a topology T on X is called <u>S - compatible</u> if ε-neigh of x ε/2-neigh of x ε/2 relation

$\forall\, x \in V \in T,\ \exists\, x \in V' \in T$ and $\mathcal{U} \in S$ s.t.

$$\mathcal{U}V' = \{ v \mid \exists\, v' \in V',\ (v,v') \in \mathcal{U} \} \subset V$$

If you extend this def to mimic the proof of Theorem 2 before we get:

Theorem 3 S is uniform on X, T S-compatible, $G \curvearrowright X$ preserving T, S, equi-cont w.r.t S; if $X^G = \emptyset$, then we get some properness for the T topology and orbits are T-closed and stabilizers are compact.

$G \overset{\text{unit}}{\curvearrowright} H$ with no inv. vectors.
Remove origin of H (s.t. there are no fixed points)

Then the orbits are <u>weak-closed</u>

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⑤ <u>Monoid</u> (semi-topological, compact)

Example: V a reflexive Banach space

$M = \{ T : V \to V \mid \|T\| \le 1 \}$ is cpt w.r.t WOT (weak op. top)
It is a semigroup/monoid as we can compose operators.

Warning: The composition $M \times M \longrightarrow M$ is NOT continuous. (The multiplication is separately continuous.)

→ This is what he means by semi-topological.

(A reflexive?) Representation of G on ~~M~~ is $G \xrightarrow{\varphi} M$

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & M \\ & \searrow & \cup \\ & & \overline{\varphi(G)} \end{array}$$

We want to study monoid compactifications.

Example: $G \cup \{\infty\}$ ← one-pt compactification

Define $x \cdot \infty = \infty \cdot x = \infty$

If $g_n \to \infty$ also $g_n^{-1} \to \infty$, and $g_n \cdot g_n^{-1} \not\to \infty$

It is not jointly continuous. So joint continuity must be ignored when ~~doing~~ working with monoids.

Fact: $\exists!$ a universal compactification.

Fix G loc cpt top Group

Look at all possible compactification

- $G \cup \infty$
- Those coming from reflexive reps

etc.

This is a classical fact: The algebra of all <u>weakly almost periodic</u> functions $\subseteq C_b(G)$.
WAP — is a closed alg.

The orbit is precompact w.r.t top.

$C_b(G)$ is a C^*-alg.

So there exists a <u>spectrum</u>

$$WAP(G) := \text{spectrum}\left(\underset{WAP}{\text{alg of}} \subseteq C_b(G)\right)$$

is the universal compactification.

<u>Theorem 4 [Veech '79]</u> (They discovered the theorem independently)

via the strategy of Howe-Moore

If G is simple Lie group, then $WAP(G) = G \cup \{\infty\}$ — the minimal object

<u>In</u> other word, any monoid compactification of G is trivial.

This follows from <u>Theorem 3</u>.

<u>Proof</u> Let M be a monoid compactification of G.

We define a uniform structure on M as follows (using continuous functions on M):

$\forall f \in C(M)$, $\varepsilon > 0$, define

$$\mathcal{U}_{f,\varepsilon} = \{(x,y) \mid \|f(\cdot x) - f(\cdot y)\|_\infty < \varepsilon\}$$
$$\cap$$
$$M \times M$$

Let S be the structure generated by these (for all f & all ε).

Theorem 3 implies $\Longrightarrow$ $M = \underbrace{G}_{\text{orbit of } M} \cup \underbrace{}_{\text{bdry}}$

bdry must consist only of fixed points for left action. **AND** right

y is fixed by left

if x, y were fixed point then $x = xy = y$

x is fixed by right

So $x = y = \infty$, one point.

G simple Lie group.

<u>Corollary</u> [Matrix coefficients of reflexive reps V are in $C_0(G)$, if $V^G = 0$.]

<u>Proof:</u> $\underset{G \cup \{\infty\}}{WAP(G)} \longrightarrow B(V)_1$

↑ closed ball of bdd ops on V

$B(V)_1 \setminus \{0\}$ has no fixed points $\Longrightarrow \infty \mapsto 0$.

By defn of matrix coefficients, they go to 0.

Howe-Moore seems to be the result that matrix coeffs go to 0...