Glueing together copies of amenable groups

Kate Juschenko

Northwestern University

Joint with Harald Helfgott

An introduction to sofic groups

Hamming distance on $S(n)$: $d(\sigma_1, \sigma_2) = |\{i : \sigma_1(i) \neq \sigma_2(i)\}|$.

Definition (Weiss)

A group Γ is sofic (Hebrew: סופי) if for every finite set $F \subseteq \Gamma$, $e \in \Gamma$ and every $\epsilon > 0$ there exist $n \in \mathbb{N}$ and a map $\phi : F \to S(n)$ such that the following conditions hold:

- $\phi(e) = id_n$,
- $d(\phi(gh), \phi(g)\phi(h)) < \epsilon$ for all g, h, such that $gh \in F$,
- $\phi(g)$ does not have fixed points, i.e. $d(\phi(g), e) = 1$, for every $g \in F \backslash \{e\}$.

We will call such a ϕ an (F, ϵ)**-approximation** of Γ.

Hamming distance on $S(n)$: $d(\sigma_1, \sigma_2) = \frac{1}{2}|\{i : \sigma_1(i) \neq \sigma_2(i)\}|$.

Definition (Gromov)

An edge colored graph $G = (V, E)$ is *initially subamenable* (or sofic) if for every $r \in \mathbb{N}$, $\varepsilon > 0$ and for every ball $B_r(G)$ of radius r in G there exists an edge-colored finite graph $G' = (E', V')$ and a finite set W in V' such that

1. G' is r-locally isometric to G. That is all r-balls $B_r(G', w)$ around every point $w \in W$ are isomorphic (as colored graphs) to $B_r(G)$.
2. W is $(1 - \varepsilon)$-large with respect to V, i.e. $|W| > (1 - \varepsilon)|V|$.

Properties of sofic groups

- A subgroup of a sofic group is sofic.
- A group is sofic if and only if all finitely generated subgroups are sofic.
- A direct product of sofic groups is again sofic.
- A direct limit of sofic groups is sofic.

Examples

- Finite groups;
- residually finite groups;
- amenable group;
- initially subamenable, i.e., for every finite set F in Γ there is an injective map ϕ from F to an amenable group such that if x, y and xy are in F then $\phi(xy) = \phi(x)\phi(y)$;
- sofic-by-amenable groups, i.e., Γ has a sofic normal subgroup N such that Γ/N is amenable;
- free products of sofic groups amalgamated over amenable subgroup.

In the perfect world where all groups are sofic...
the following are true:

Connes' embedding problem:
Every II_1-factor embeds into $\mathcal{R}^\omega$

Kaplanski's direct finiteness conjecture:
Let K be a field and Γ be a group (can be assumed finitely generated). Does the equality $ab = e$ implies $ba = e$ for every a, b in $K[\Gamma]$?

Fuglede-Kadison determinant conjecture:
$\ln\det(\Lambda) \geq 0$ for every positive $\Lambda \in M_d(\mathbb{Z}\Gamma) \subset M_d(L\Gamma)$.

Let N be a von Neumann algebra with a faithful normal trace τ. The spectral density function associated to a positive operator $\Delta \in N$ is a function $F_\Delta : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$F_\Delta(\lambda) = \tau(\chi_{[0,\lambda]}(\Delta)).$$

Fuglede-Kadison determinant defined by

$$\ln\det_N(\Delta) = \begin{cases} \int\limits_{0+}^{+\infty} \ln(\lambda) dF_\Delta(\lambda), & \text{if the integral converges,} \\ -\infty, & \text{otherwise.} \end{cases}$$

Definition

A discrete group G is *surjunctive* if, for every $k \in \mathbb{N}$, if one considers the left shift action $G \curvearrowright \{1, \ldots, k\}^G$ then every continuous G-equivariant injective map from $\{1, \ldots, k\}^G$ to itself is surjective.

Gottschalk's surjunctivity:
Is every countable discrete group surjunctive?

Does there exists a non-sofic group?

Higman group

H_4 is the group generated by elements a_1, a_2, a_3, a_4 subject to the following relations:

$$a_1^{-1} a_2 a_1 = a_2^2,$$
$$a_2^{-1} a_3 a_2 = a_3^2,$$
$$a_3^{-1} a_4 a_3 = a_4^2,$$
$$a_4^{-1} a_1 a_4 = a_1^2.$$

Composed from Baumslag-Solitar group

$$BS(1,2) = \langle a, b : b^{-1} ab = a^2 \rangle$$

Why Higman group?

Theorem (Higman)

Every homomorphism of H_4 into a finite group is trivial.

Consider the following semidirect product:

$$H_4 \rtimes \mathbb{Z}/4\mathbb{Z}$$

by letting $\mathbb{Z}/4\mathbb{Z}$ act on H_4 as follows: we choose a generator t of $\mathbb{Z}/4\mathbb{Z}$, and we let $t(a_i) = a_{i+1}$ for $i = 1, 2, 3$, $t(a_4) = a_1$.

$$H_4 \rtimes \mathbb{Z}/4\mathbb{Z} = \langle a, t : b^{-1}ab = a^2, b = t^{-1}at \rangle.$$

Simplified argument of Higman:

$|a| = |b| = n$, $b^{-n}ab^n = a^{2^n}$, thus n divides $2^n - 1$ which is a contradiction.

Let G be a group. *An invariant length function* is a map $l : G \to [0, 1]$ such that $l(g) = 0$ if and only if $g = e$, and

$$l(gh) \leq l(g) + l(h), l(g^{-1}) = l(g), \text{ and } l(gh) = l(hg),$$

for all $g, h \in G$.
It is called *commutator contractive* if

$$l([g, h]) \leq 4l(g)l(h), \text{ for all } g, h \in G.$$

Hamming metric is not commutator contractive. Example of commutator contractive metric: $\frac{1}{2}\|1 - g\|, g \in U(H)$.

Theorem (A. Thom, '10)

Higman's group does not embed into a metric ultraproduct of finite groups with a commutator-contractive invariant length function.

Higman group is SQ-universal, i.e., every countable group can be embedded in one of its quotient groups.

Theorem (Helfgott-Juschenko)

Higman group H_4 *is sofic then for any* $\epsilon > 0$*, there is an* p *and a bijection* $f : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ *such that*

$$f(x+1) = 2f(x) \tag{1}$$

for at least $(1-\epsilon)n$ *elements* x *of* $\mathbb{Z}/p\mathbb{Z}$ *and*

$$f(f(f(f(x)))) = x \tag{2}$$

for all $x \in \mathbb{Z}/p\mathbb{Z}$*.*

Sofic approximations of amenable groups

Let G be an amenable group generated by a finite symmetric set S, and

$$e \in F_1 \subset F_2 \subset \ldots \subset F_k \subset \ldots \subset G$$

such that $|sF_j \Delta F_j| \leq |F_j|/j$ for all $s \in S$ and all $j \geq 1$.
In addition, we assume that given any $\eta > 0$ we have

$$|(F_{j-1}^{-1} F_j) \setminus F_j| \leq \eta |F_j|$$

for all $j > 1$, simply by replacing $\{F_j\}_{j \geq 1}$ by a subsequence.

Lemma (Analog of Ornstein-Wiess, Kerr-Li)

For any $\epsilon, \kappa > 0$*, there are* $k \geq 1$ *and* $\lambda_1, \ldots, \lambda_k \in (0, 1]$ *with* $1 - \epsilon \leq \lambda_1 + \cdots + \lambda_k \leq 1$ *such that the following holds. For any infinite sequence of finite subsets*

$$e \in F_1 \subset F_2 \subset \ldots \subset F_k \subset \ldots \subset G$$

satisfyingthe above for $\eta = 1$*, there are* $\delta > 0$*,* $N \geq 1$ *and a finite set* $S \subset G$ *such that, if* $\phi : S \to Sym(n)$ *is an* (S, δ, n)*-sofic approximation with* $n \geq N$*, there exist* $C_1, \ldots, C_k \subset \{1, \ldots, n\}$ *such that*

1. *the sets* $\phi(F_1)C_1, \ldots, \phi(F_k)C_k$ *are pairwise disjoint,*
2. *for every* $1 \leq j \leq k$ *and every* $c \in C_j$*, the map* $s \mapsto \phi(s)c$ *from* F_j *to* $\phi(F_j)c \subset \{1, \ldots, n\}$ *is injective,*
3. *the family* $\{\phi(F_j)c\}_{1 \leq j \leq k,\, c \in C_j}$ *is* ϵ*-disjoint and* $(1 - \epsilon)$*-covers* $\{1, \ldots, n\}$*,*
4. $(1 - \kappa)\lambda_j \leq |\phi(F_j)C_j|/n \leq (1 + \kappa)\lambda_j$ *for every* $j = 1, 2, \ldots, k$*.*

Følner sets of $BS(1,2)$:
$F_n = \{b^i a^j : 0 \le i \le n, 0 \le j \le 2^n\}$

Lemma (Rokhlin)

Let T be a measure-preserving ergodic transformation on $\mathbb{R}/\mathbb{Z}$. Then, for any $k > 0$ and any $\epsilon > 0$, there exists a measurable set E such that $\mu(T^{-i}(E) \cap T^{-j}(E)) = 0$ for all $0 \le i < j < k$ and $\mu(\cup_{0 \le i < k} T^{-i}(E)) > 1 - \epsilon$.

It is sufficient to prove that we can re-numerate m disjoint copies of F_n so that the action of ϕ will have the form

$$\phi(a)(x) = x + 1 \mod n, \quad \phi(b)(x) = x/2 \mod n$$

at $(1-\epsilon)$-every $x \in \{0, \dots, n-1\}$
Apply Rohklin lemma to $T : x \mapsto 2x$.

Obs: If $I \subset [0, n)$ is a (closed-open) subinterval, then $T^{-1}(I)$ is a union of two disjoint subintervals $I/2$ and $I + n/2$

Theorem (Helfgott-Juschenko)

Higman group H_4 is sofic then for any $\epsilon > 0$, there is an p and a bijection $f : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ such that

$$f(x+1) = 2f(x) \tag{3}$$

for at least $(1-\epsilon)n$ elements x of $\mathbb{Z}/p\mathbb{Z}$ and

$$f(f(f(f(x)))) = x \tag{4}$$

for all $x \in \mathbb{Z}/p\mathbb{Z}$.

Using methods of p-adic analysis of Holdman-Robinson we obtained:

Theorem

Let a bijection $f : \mathbb{Z}/3^r\mathbb{Z} \mapsto \mathbb{Z}/3^r\mathbb{Z}$*,* $r \geq 1$ *be given. Then*

$$\textit{either} \;\; f(x+1) \neq 2f(x) \quad \textit{or} \quad f(f(f(f(x)))) \neq x \tag{5}$$

for at least $3^{r/4-1}$ *values of* $x \in \mathbb{Z}/3^r\mathbb{Z}$*.*

Thank you!