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Super intrinsic synthesis in fixed point properties

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Consider a class $\mathcal{X}$ of metric spaces
G a f.g. group (sometime also consider $M \leq G$ subgr.)

$X \in \mathcal{X}$ and $G \curvearrowright X$ by isometries

__Goal:__ Study a fixed point property $F_{\mathcal{X}}$ for $\mathcal{X}$ of "unbounded wildness"
__Note:__ It's hard to solve fixed point properties..

(1) consider a relative fixed point property $\mathrm{Rel}(F_{\mathcal{X}})$
(2) Under what conditions $\boxed{\text{relative fix} \Rightarrow F_{\mathcal{X}}}$?

__Def:__ (1) $G \geq M$ has $\underline{\mathrm{rel}(F_{\mathcal{X}})}$ $\overset{\text{def}}{\iff}$

$$\forall X \in \mathcal{X}, \forall G \curvearrowright X, \; X^M \neq \emptyset$$

(2) G has $(F_{\mathcal{X}}) \underset{\text{def}}{\iff} G \geq G$ rel$(F_{\mathcal{X}})$

__Rem__ If we consider inclusions of subgroup then
$G \geq G_0 \geq M_0 \geq M$ and $G_0 \geq M_0$ has rel$(F_{\mathcal{X}})$,
Then $G \geq M$ has rel$(F_{\mathcal{X}})$, from definition.

(specific)
<u>Example</u>: $R \ni 1$ f.g. generated associative ring

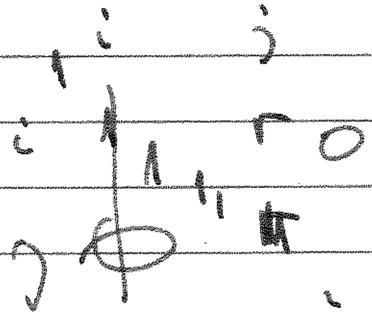
<u>Exp</u>: $\mathbb{Z}$, $\mathbb{F}_p[\mathbb{F}_2] = \mathbb{F}_p\langle t_1^{\pm}, t_2^{\pm}\rangle$
↑ finite field

Let $n \geq 3$

$G_R := E(n, R)$ elementary groups

$:= \langle e_{ij}(r) \mid i \neq j \in \{1, \dots, n\},\ r \in R \rangle$

(group generated by elementary matrices)



G_R

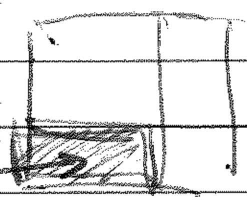
$\vee\!\!/$

M_R $\begin{bmatrix} I_{n-1} & * \\ 0 & 1 \end{bmatrix}$

G_R

$\vee\!\!/$

L_R →



☆

<u>Theorem</u>: In ☆,

(1) [KASSABOV, '07]

for $n \geq 3$ $G_R \geq M_R$ have rel$(F*)$

$G_R \geq L_R$

(2) [BADER, FURMAN, GELANDER, MONOD
OLIVIER, M , '11]

$\forall n \geq 4$ — $rel(F$ $B_{NCL_q})$ $\forall q \in [1,\infty)$

— — $rel(F$ $QB_{L_q})$ ~~for~~ almost all $q \in [1,\infty)$
except $\frac{4}{3}, \frac{6}{5}, \frac{8}{7}, \ldots$

For $q \in (1,\infty)$

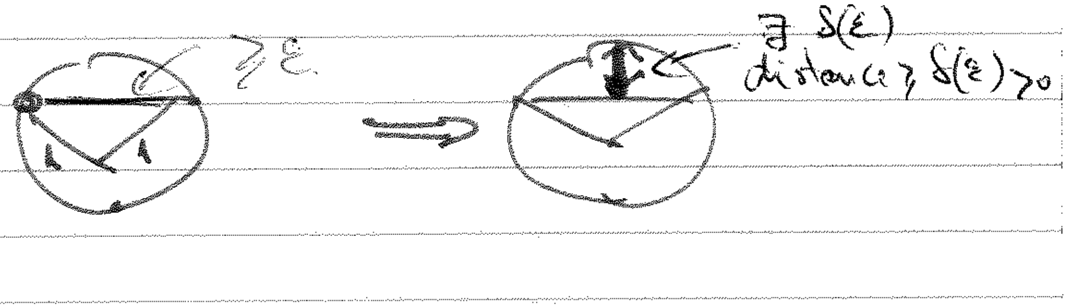
B_{NCL_q} = all non-commutative L_q-spaces
(or just L_q spaces)

QB_{L_q} := quotient spaces of L_q-spaces

Recall the goal:
"Unbounded wildness"
means getting close to
either 1 and ∞.

1 — 2 — ∞
our best friend

B_{UC} = set of unif. convex Banach spaces



1. Who cares?

[illegible]

Note $F_{Hilb} \iff$ property (T)

(1) **Theorem** (Yu, Nica, Alvarez + V. Lafforgue)

$\forall H$ infinite hyperbolic

$\exists\ 1 < q_1 \leq 2 \leq q_2 < \infty$ s.t.

$\forall q \in (1, q_1) \cup (q_2, \infty)$ then $\exists$

metrically proper isometric H-action on

$\mathcal{B}_{QL_q} \leftarrow$ quotients of L_q spaces

(2) Super expanders

These are like expanders where the maps go to Banach spaces (instead of Hilbert spaces)

unif convex

$\mathcal{B}_{uc}$

$$G \qquad F_{Buc}$$

$G \swarrow G_1 \quad \searrow G_2 \ldots\ldots$

$|G_n| < \infty$

Then $(G_n)_n$ form a super expander.

$E(n, R)$

Q: (unbounded rank expanders)

$SL(4, \mathbb{Z}) \longleftarrow$ has (T) or F_{Hilb}

$\downarrow$

$SL(4, \mathbb{F}_{p_n})$ p_n prime

These cannot form an expander.

Q: Can $(SL(4m, \mathbb{F}_{p_m}))_m$ form expanders?

A: Yes, by Kassabov, uses F_{Hilb}.

2. Synthesis: $Rel(F_X)_s \Longrightarrow F_X$

We want to synthesize Theorem 1.

(1) {ERSHOV - JAIKIN '10} considered "ε-orthogonality"

Kassabov $\searrow$ ZAPIRAIN

$\searrow$ E-J-K

OPPENHEIM, LAVY

intrinsic synthesis

<u>Theorem 2</u> (E-J '10)
$\forall\, n \geq 3$ GR has $\left(F_{Hilb}\right)$
$\Downarrow$
property T.

(2) Next synthesis is by Shalom
(1st intrinsic synthesis)

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some comment to the expander Q: $SL(4m, \mathbb{F}_{p^n}) \simeq E(4, Mat_{m\times m}(\mathbb{F}_{p^n}))$

<u>Theorem</u> (Shalom 1999, Pub IHES)
$G \geq M_1 \ldots M_l$
X st. "$F_X \Rightarrow \forall\, G \curvearrowright X$, G-orbits are bdd."
↪ exp: reflexive Ban, B_{NCL_1}) ....

(BG) hypothesis: $U := \cup M_i$
bounded generation — boundedly generate G

$\exists N$ s.t.
$\forall g \in G$, $g = u_1 \ldots u_N$, for some $u_j \in U$
but $\underline{N \text{ is fixed}}$ (bdd)

Then $G \geq M_i$ rel (F_X) $\forall i$ $\Longrightarrow$ G has (F_X).

(BG) is very strong
?. F_2 does not have (BG)?

(BG) is a strong condition...

3. Main result

Natural question (Shalom 1999)
Can one establish super intrinsic synthesis ?
without assuming (BG)

Theorem (M): YES!

Cor 3 (M) In ⊛

(1) (E-2) $\forall n \geq 3$, G_R has F_{tree}

(2) $\forall n \geq 4$, G_R has F_{BNCL_ξ} $\forall \xi \in [1, \infty)$

F_{QBL_ξ} for almost all $\xi \in (1, \infty)$

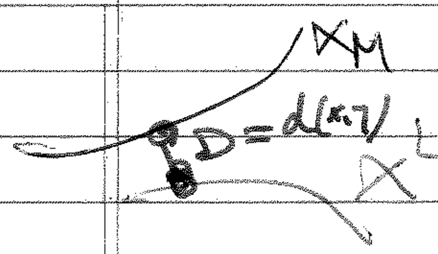
4. Heart of proof (inspired by Shalom's proof from '06 ICM proceedings)

We stick to ★: $G := G_R$
$M := M_R$
$L := L_R$

$G \curvearrowright X$ metric space s.t. $X^M \neq \emptyset$ and $X^L \neq \emptyset$.

We want to show $X^G \neq \emptyset$ (a global fix point).

<u>Assumption 4</u>: $\exists!\ (x,y) \in X^M \times X^L$ s.t. $D := \mathrm{dist}(X^M, X^L)$

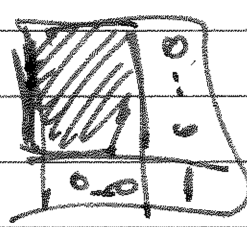


<u>Key Proposition</u> (self-improvement argument)
Under assumption 4, $x \in X^G$.

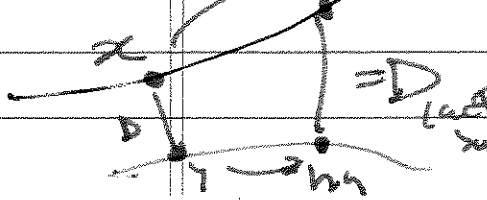
<u>Proof</u>: Take $H_1, H_2 \leq G$ "variables" st $\underbrace{x \in X^{H_1}}_{x \text{ is fixed by } H_1}$ and $y \in X^{H_2}$.

initially

H_1	H_2
M	L

1st move: Take $P = $  $(\simeq E(n-1, R))$

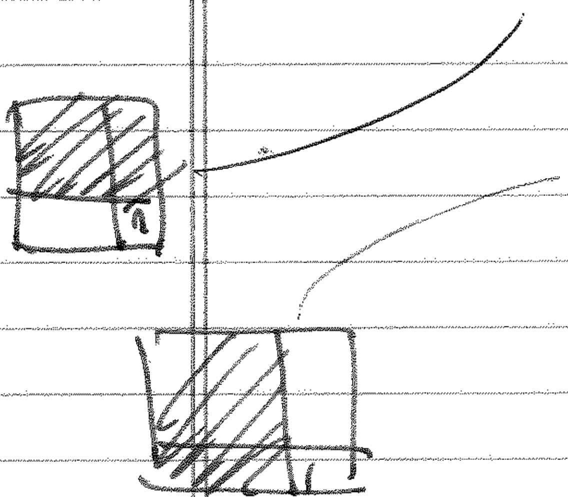
$\forall h \in P$, $hH_1h^{-1} \geq M$
$hH_2h^{-1} \geq L$



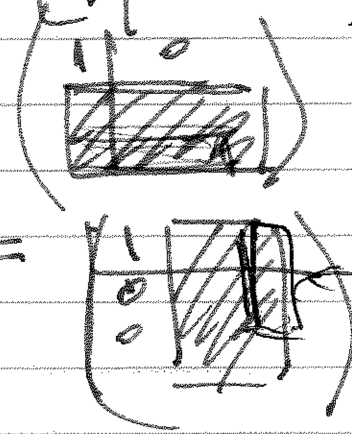
$\longrightarrow x \in X^{\langle H_1, P \rangle}$

Enlargement:

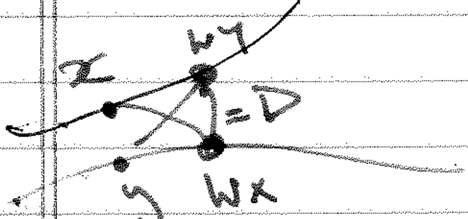
H_1	H_2
$\langle H_1, P\rangle$	$\langle H_2, P\rangle$



2nd word: Take $W = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-2} & 0 \\ 1 & 0 & 0 \end{pmatrix} \in G$

Miracle: $wH_1W^{-1} =$  $\supseteq L$

$wH_2W^{-1} =$ $\supseteq M$



Finally, we enlarged as

H_1	H_2
$\langle H_1, wH_2w^{-1}\rangle$	$\langle H_2, wH_1w^{-1}\rangle$

$G' :=$ (left), $=: G$ (right)

arXiv: 1505:06728v2 ← full version

arXiv: 1611:00337 on Thm 2 [8 pages.]