Lp-compression of wreath-products and some related groups

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Joint work with Jérémie Brieussel

Plan

- $\alpha_p^{\#}(H \wr \mathbb{Z})$ expressed in terms of $\alpha_p^{\#}(H)$, for $p \in [1,2]$
- by "scaling", construct elementary amenable groups with prescribed compression function (analogue of Arzhantseva-Druțu-Sapir)
- examples of 3-step solvable groups satisfying
$$\alpha_p^{\#}(G) > \alpha_2^{\#}(G) \quad \text{for } p > 2$$

Def: (Guentner - Kaminker)
Given two metric spaces (X, d_X) and (Y, d_Y), define the compression exponent

$$\alpha_Y^*(X) = \sup\left\{\alpha \;\middle|\; \begin{array}{l} \exists \text{ 1-Lipschitz map } f: X \to Y \text{ s.t.} \\ d_Y(f(x_1), f(x_2)) \geq c \cdot d_X(x_1, x_2)^{\alpha} \\ \text{for some } c \end{array}\right\}$$

Restrict to $(X, d_X) = (G, d_S)$ and $Y = L_p([0,1])$

We can also define the <u>equivariant compression exponent</u>:

$$\alpha_p^{\#}(G) = \sup\left\{ \alpha \;\middle|\; \begin{array}{l} \exists \text{ 1-Lipschitz and equivariant} \\ \text{map } f: G \to L_p \text{ s.t.} \\ \quad \|f(x_1)-f(x_2)\|_p \geq c \cdot d(x_1,x_2)^{\alpha}, \text{ for some } c \\ \text{and } \|f(gx_1)-f(gx_2)\|_p = \|f(x_1)-f(x_2)\|_p \end{array} \right\}$$

<u>Remarks</u>: (1) The first compression exponent is qi-invariant.
(2) For amenable groups, $\alpha_p^{*}(G) = \alpha_p^{\#}(G)$.
(The case $p=2$ is due to Gromov; general p to Naor-Peres.)

<u>Some known results</u>

- Bourgain
  For $\mathbb{F}_k$, $k \geq 2$, $\alpha_2^{*}(\mathbb{F}_k) = 1$ and $\alpha_2^{\#}(\mathbb{F}_k) = \frac{1}{2}$.

- Tessera
  For $\left\{ \begin{array}{l} \text{polycyclic groups} \\ (\text{finite}) \wr \mathbb{Z} \\ \text{Baumslag-Solitar} \end{array} \right\}$ $\quad \alpha_p^{\#}(G) = 1$

- Arzhantseva - Guba - Sapir
  For F the Thompson's group:
  $\alpha_2^{*}(F) = \alpha_2^{\#}(F) = \frac{1}{2}$; also $\frac{1}{2} \leq \alpha_2^{\#}(\mathbb{Z} \wr \mathbb{Z}) \leq \frac{3}{4}$.

- Austin - Naor - Peres: $\quad \alpha_2^{\#}(\mathbb{Z} \wr \mathbb{Z}) = \frac{2}{3}$

- Naor - Peres: $\quad \alpha_p^{\#}(\mathbb{Z} \wr \mathbb{Z}) = \max\left\{ \frac{p}{2p-1}, \frac{2}{3} \right\}$.

Theorem (Brieussel - Z).
Let H be an infinite finitely generated group and $p \in [1,2]$. Then

$$\alpha_p^{\#}(H \wr \mathbb{Z}) = \min\left\{ \alpha_p^{\#}(H), \ \frac{\alpha_p^{\#}(H)}{\alpha_p^{\#}(H) + \left(1 - \frac{1}{p}\right)} \right\}$$

Recall: $H \wr \mathbb{Z} = \left(\bigoplus_{\mathbb{Z}} H \right) \rtimes \mathbb{Z}$

For $f, f' : \mathbb{Z} \to H$ and $z, z' \in \mathbb{Z}$, the multiplication is given by
$$(f, z)(f', z') = (f \, \tau_z f', \ z + z').$$

Starting point: Naor - Peres
Markov type inequality (Pisier martingale inequality)
Let L_p, $p \in [1,2]$.
Choose any symmetric probability measure μ on G.
$X_1, X_2, \ldots$ i.i.d. $\sim \mu$
The random walk $W_n = X_1 X_2 \ldots X_n$.

Let $f : G \to L_p$ be a 1-cocycle. It behaves very much like martingales:

$$\mathbb{E} \, \|f(W_n)\|_p^p \leq C_p \cdot n \cdot \underbrace{\mathbb{E} \, \|f(W_1)\|_p^p}_{\leq C \cdot \sum_{g \in G} |g|^p \cdot \mu(g)}$$

Introduce:

$$\beta_p^*(G) = \sup\left\{ \lim_{n\to\infty} \frac{\log E|W_n|}{\log n} \;\middle|\; \mu \text{ with finite } p\text{-moment} \right\}$$

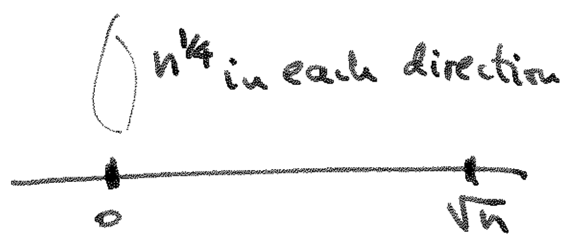
For $\mathbb{Z}$ and simple random walk (RW), $\beta_2^*(\mathbb{Z}) = \frac{1}{2}$

For $\mathbb{Z}$ and α-stable RWs, $\mu(z) \sim \frac{1}{|z|^{1+\alpha}}$, $1 \le \alpha < 2$,

and $\beta_p^*(\mathbb{Z}) = \frac{1}{p}$; for $p \in [1,2]$

Naor-Peres proved that $\alpha_p^*(G) \le \dfrac{1}{p \cdot \beta_p^*(G)}$.

Note: For $\mathbb{Z} \wr \mathbb{Z}$, $\beta_2^*(G) = \frac{3}{4}$



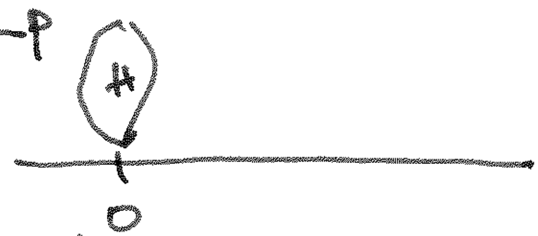
More general lamplighter groups.

Proof of $\alpha_p^{\#}(H \wr \mathbb{Z})$

Note first that it is the case that $\alpha_p^{\#}(H \wr \mathbb{Z}) \le \alpha_p^{\#}(H)$.

Look at finite subsets. $\alpha = \alpha_p^{\#}(H) + \varepsilon$

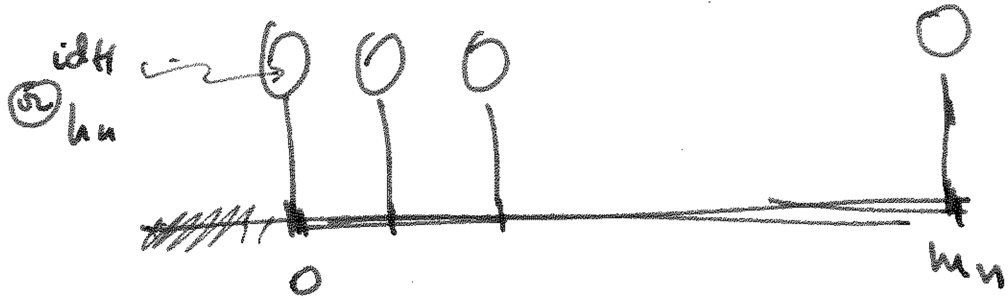
$\Phi : H \wr \mathbb{Z} \longrightarrow L_p$



Look at the restriction to $H \times \{0\}$.

(h_n), $|h_n| \geq n$, $\|\Phi(h_n)\|_p \leq C \cdot n^2$.

$$\mathcal{G}_n = \left\{ (f, z) \;\middle|\; \begin{array}{l} \operatorname{supp}(f) \subseteq [0, m_n],\ z \in [0, m_n], \\ f(x) \in \{ id_H, h_n \} \end{array} \right\}$$



Consider $\gamma \in [1,2]$. Move on $[0, m_n]$: $P_n(z_1, z_2) \sim \dfrac{c}{|z_1 - z_2|^{1+\gamma}}$

Let U randomize $\{ id_H, h_n \}$ and let $Q_n = U P_n U$.

<u>Revelle-Peres:</u>

$$\sum_{u,v \in \mathcal{G}_n} |\Phi(u) - \Phi(v)|^2 \cdot \pi_n(u) \cdot \pi_n(v) \leq$$

$$\leq C(n) \cdot \sum_{u,v} |\phi(u) - \phi(v)|^2 \cdot \pi_n(u) \cdot Q_n(u,v)$$

$$\text{where } C(n) = \begin{cases} m_n^{\gamma} & , \gamma \in (1,2] \\ m_n \cdot \log(m_n) & , \gamma = 1 \\ m_n & , \gamma < 1 \end{cases}$$

Same inequality holds true when using Lp-norms:

$$(*) \qquad \sum_{u,v \in \mathcal{G}_n} \|\Phi(u) - \Phi(v)\|_p^p \cdot \pi_n(u) \cdot \pi_n(v) \leq C(n) \cdot \sum_{u,v} \|\Phi(u) - \Phi(v)\|_p^p \cdot \pi_n(u) \cdot Q_n(u,v) \quad (*)$$

We know that the sum on the right hand side can be estimated:

$$\underbrace{\|\Phi(h_n)\|_p^p}_{(\mathrm{I})} + \underbrace{\sum_{u,v} |u-v|^p \cdot P_n(u,v) \cdot \frac{1}{m_n}}_{(\mathrm{II})}$$

Here: $(\mathrm{I}) \leq \varepsilon \cdot n^{\alpha \cdot p}$

Choose m_n and γ (the optimal choice is $\gamma = 1$) such that (I) and (II) and extract distortion estimate from the Poincaré inequality $(*)$.

Question: What compression exponents can elementary amenable groups admit?

Austin: There exists G solvable s.t. $\alpha_p^{\#}(G) = 0$, $\forall p \in (1,\infty)$.

Olshanskii - Osin (General Embedding Theorem)

$\Gamma = \langle A, B \rangle$, A and B finite subgroups of Γ.

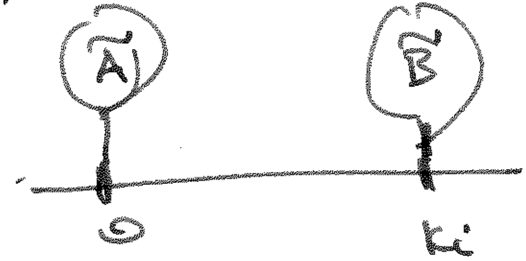
$\Gamma \to \Gamma_i$ a sequence of quotients, $\Gamma_i = \langle \dot{A}, \dot{B} \rangle$.

Look at $\bigoplus_i \Gamma_i$ with a length function satisfying $\ell(u_i) = k_i \cdot |u_i|_{\Gamma_i}$ and $k_i \xrightarrow{i} \infty$.

We want to construct a finitely generated group G such that $(\bigoplus \Gamma_i, \ell) \hookrightarrow G$.

Consider $G_i = \Gamma_i \wr \mathbb{Z}$, marked with $(t, \tilde{A}, \tilde{B})$ ⑦

Here $t = (id, t)$

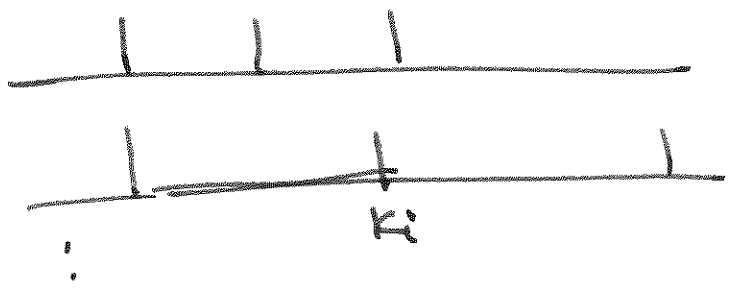


Take

$$G = \bigotimes_i G_i = (\mathbb{Z} * A * B) \Big/ \bigcap_i \ker(\mathbb{Z} * A * B \to G_i)$$

$$\Big(G_i \longrightarrow \underbrace{(A \times B) \wr \mathbb{Z}}_{\text{"global consistency"}} \Big)$$

Metric on G is tractable ...



<u>Examples</u>: (1) (Γ_i) expanders $\leftarrow$ analogous to ADS

(2) $\Gamma = D_\infty = \langle a, b \mid a^2 = b^2 = 1 \rangle$

$\Gamma_i = D_{2l_i}$

$[l_i]_{k_i}^{\infty}$ obstruction from metric cotype

[Mendel-Naor]

<u>Question</u>: We were able to evaluate precisely $\alpha_p^{\#}(G)$ when G was constructed with dihedral groups. This is larger than $\alpha_2^{\#}(G)$.

<u>Q</u>: What $\alpha_2^{\#}(G)$ can be achieved by finitely presented groups?