On spectra of Koopman, groupoid and quasi-regular representations.

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Amenability, coarse embeddability and fixed points properties

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1 Hecke type operators

G – countable group

$\rho : G \longrightarrow U(\mathcal{H})$ – unitary representation

Hilbert space

$m = \sum_g c_g \, g \in \mathbb{C}[G]$ – element of

group algebra

$$M_m = \rho(m) = \sum_g c_g \rho(g)$$ - bounded operator (self adjoint if $m^* = m$)

↑ Hecke type operator

Popular question: $sp(M_m) = ?$

Example: if G is torsion free and for some $m \in \mathbb{C}[G]$ and regular representation the spectrum of M_m has gap $\Rightarrow$ Kadison-Kaplanski Conjecture is wrong.

② Three types of popular representations.

a) Regular and quasiregular representations

λ_G, $\ell^2(G)$ — regular

$\lambda_{G/H}$, $\ell^2(G/H)$, $G/H = \{gH : g \in G\}$

coset set

$G \curvearrowright G/H$ — quasiregular repr. ($\Leftrightarrow$ permutational representation).

$G \curvearrowright X$, $\forall x \in X$,

$$\rho_x : G \to U(\ell^2(Gx))$$

orbit

$$\rho_x \approx \lambda_{G/G_x}, \quad G_x = \mathrm{St}_G(x)$$ - stabilizer

unitary equivalence

(G, X, μ) $\quad \mu$ – invariant (or quasi-invariant) measure

$\{\rho_x\}_{x\in X}$ – family of quasi-regular representations.

Q. Given $m \in \mathbb{C}[G]$, how $sp(\rho_x(m))$ depends on $x \in X$?

b) Koopman representation

(G, X, μ) μ – quasi-invariant probability measure.

$$K : G \to U(L^2(X,\mu))$$

$$(K(g)f)(g) = \sqrt{\frac{d\mu(g^{-1}x)}{d\mu(x)}}\, f(g^{-1}x)$$

the Radon-Nikodim deriv.

$$(\kappa(g)f)(g) = f(g^{-1}x) \text{ if } \mu \text{ is invariant.}$$

c) Groupoid representation.

$$(G, X, \mu)$$

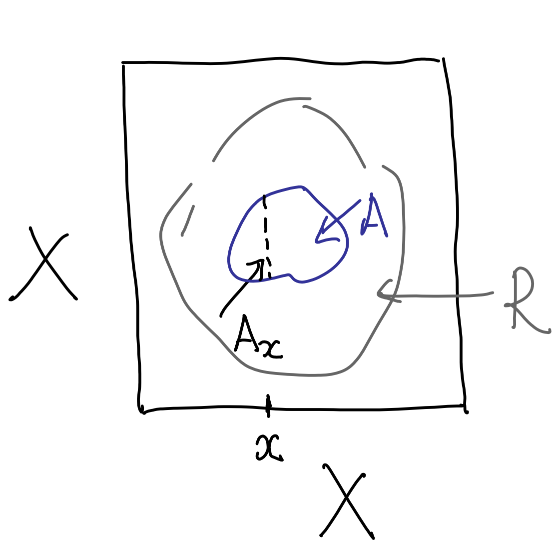
$X \times X \supset R$ – orbit equivalence relation

$$R = \{(x,y) \mid x, y \in X, \exists g \in G, y = g(x)\}$$

ν — measure on $\mathcal{R}$

For $A \subset \mathcal{R}$

$$\nu(A) = \int_X |A_x| \, d\mu(x)$$



$\nu = \mu \times$ counting measure

$$\pi : G \to U(L^2(\mathcal{R}, \nu))$$

$$\big(\pi(g) f\big)(x,y) = f(g^{-1}x, y). \text{ — groupoid repres.}$$

$$\boxed{\pi \approx \int_X \rho_x \, d\mu(x)}$$ Main result:

④ Theorem. [Artem Dudko, R. Gri...]

1) For an ergodic action (G, X, μ) where

μ is a probability quasi-invariant measure and any $m \in \mathbb{C}[G]$ one has

$$sp(K(m)) \supset sp(\rho_x(m)) = sp(\pi(m))$$

for μ-almost all $x \in X$.

2) if, moreover, μ is G-invariant and non-atomic, then $sp(K_0(m)) \supset sp(\pi(m))$, where K_0 is the restriction of K onto the orthogonal complement to

constant functions.

3) if, in addition to the conditions of 1),

(G, X, μ) is hyperfinite (i.e. Zimmer-amenable),

then

$$\mathrm{sp}(K(m)) = \mathrm{sp}(\pi(m)).$$

Reformulation on the language of weak containment of representations.

ϱ – unitary repr. of G

C_ϱ – C^*-algebra generated by operators $\varrho(g)$, $g \in G$

$\varrho \prec \eta$ iff $\varphi : C_\eta \twoheadrightarrow C_\varrho$ – surjective homomorphism of C^*-algebras

weak containment

Th. [Hulanicki 1966] TFAE

(i) G is amenable

(ii) $1_G \prec \lambda_G$

↑ trivial repres.

(iii) $\rho \prec \lambda_G$ for every unitary representation ρ of G

Part 2) of theorem means that

$$K \succ \rho_x \sim \pi$$

weak equivalence

for μ-almost all $x \in X$.

And 3) means that

$$K \sim \pi \quad (\sim \rho_x \quad - \mu - \text{almost sure})$$

"random" quasi-regular repr.

5 Spectra of groups and graphs

$\Gamma = (V, E)$ graph, $f \in \ell^2(V)$

$$(Mf)(x) = \frac{1}{\deg x} \sum_{y \sim x} f(y), \quad \boxed{sp(M) \subset [-1, 1]}$$

Markov operator

incidence relation

$\Gamma(G, S)$ – Cayley graph

system of generators $S = \{s_1, \dots, s_k\}$

$\Gamma(G, H, S)$ - Schreier graph

$G > H$ - subgroup

$V = \{ gH \mid g \in G \}$

vertices

$gH \xrightarrow{s} sgH$

edges

$w : S \cup S^{-1} \longrightarrow \mathbb{R}$ - weight function

M_w - weighted Markov operator

if w is identified with

$$m = \sum_{\substack{\varepsilon=\pm 1 \\ i}} w(s_i^{\varepsilon})\, s_i^{\varepsilon} \in \mathbb{R}[G]$$

then $\boxed{sp(M_w) = sp(\rho(m))}$ where

ρ is a regular or a quasiregular representation

$M = \int \lambda \, dE(\lambda)$ - spectral decomposition

$V \ni v \qquad \delta_v \in \ell^2(V)$

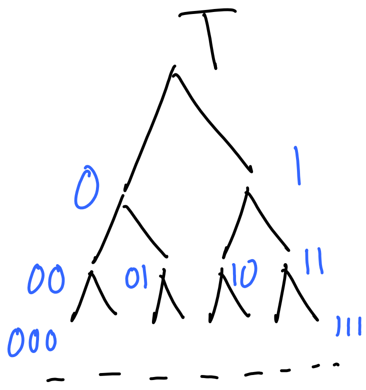
delta mass at v

μ_v - spectral measure, $\quad \mu_v(B) = \langle E(B)\delta_v, \delta_v \rangle$

$$(\Gamma_n, v_n) \longrightarrow (\Gamma, v) \Rightarrow \mu_{v_n} \xrightarrow{\text{*-weakly}} \mu$$

convergence of marked graphs

(6) Groups acting on rooted trees



$G \le \mathrm{Aut}(T)$

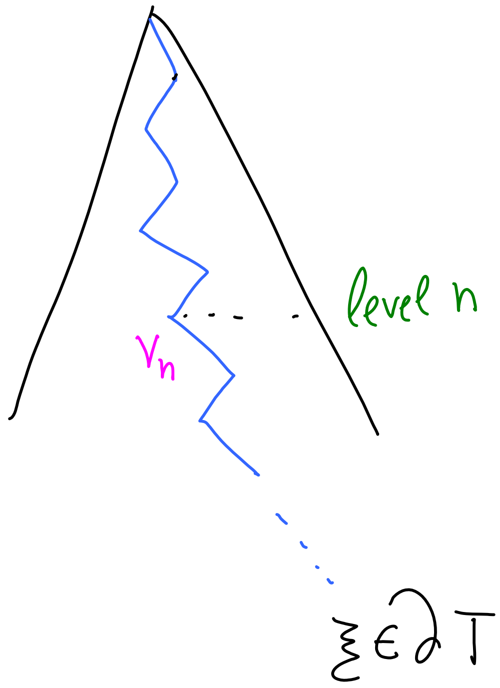
$G \curvearrowright \partial T$

$(G, \partial T, \mu)$

$\partial T = \{0,1\}^{\mathbb{N}}$ – Boundary

uniform Berroulli measure $\mu = \{\frac{1}{2}, \frac{1}{2}\}^{\mathbb{N}}$

$$\partial T \ni \xi = \{v_n\}_{n=0}^{\infty}$$



Γ_n – graph of action on level n

Γ_ξ – graph of action on orbit $G\xi$

$$(\Gamma_n, v_n) \longrightarrow (\Gamma_\xi, \xi)$$

K - Koopman representation in $L^2(\partial T, \mu)$

ρ_n - permutational representation of G in $\ell^2(L_n)$

level n

Th. [Bartholdi, Gri.., 2000]

a) For each self-adjoint $m \in \mathbb{C}[G]$

$$\boxed{sp(K(m)) = \overline{\bigcup_{n \geq 0} sp(\rho_n(m))}}$$

(b) For each $\xi \in \partial T$

$$\boxed{sp(\rho_{G/G_\xi}(m)) \subset sp(K(m))}$$

and if the graph Γ_ξ is amenable $(\Leftrightarrow$ $(G, G/G_\xi)$ is amenable$)$, then

$$\boxed{sp(\rho_{G/G_\xi}(m)) = sp(K(m))}$$

c) if the subgroup G_ξ is amenable, then

$$\boxed{sp(\rho_{G/G_\xi}(m)) \subset sp(\lambda_G(m)).}$$

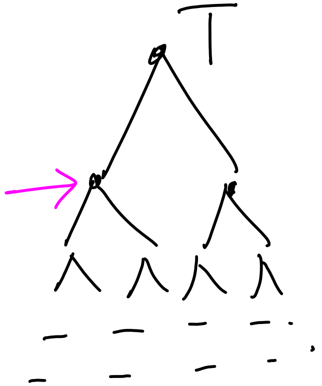
regular repres.

Prop. [Bar..., Gr. 2000]
Let G be a torsion-free amenable group with the finite generating set $S = S^{-1}$ such that there is a homomorphism $\varphi: G \to \mathbb{Z}/2\mathbb{Z}$ with $\varphi(S) = \{1\}$

Then $\quad sp(M) = [-1, 1]$.

$$M = \frac{1}{|S|} \sum_{s \in S} \lambda_G(s)$$ — Markov operator of simple random walk

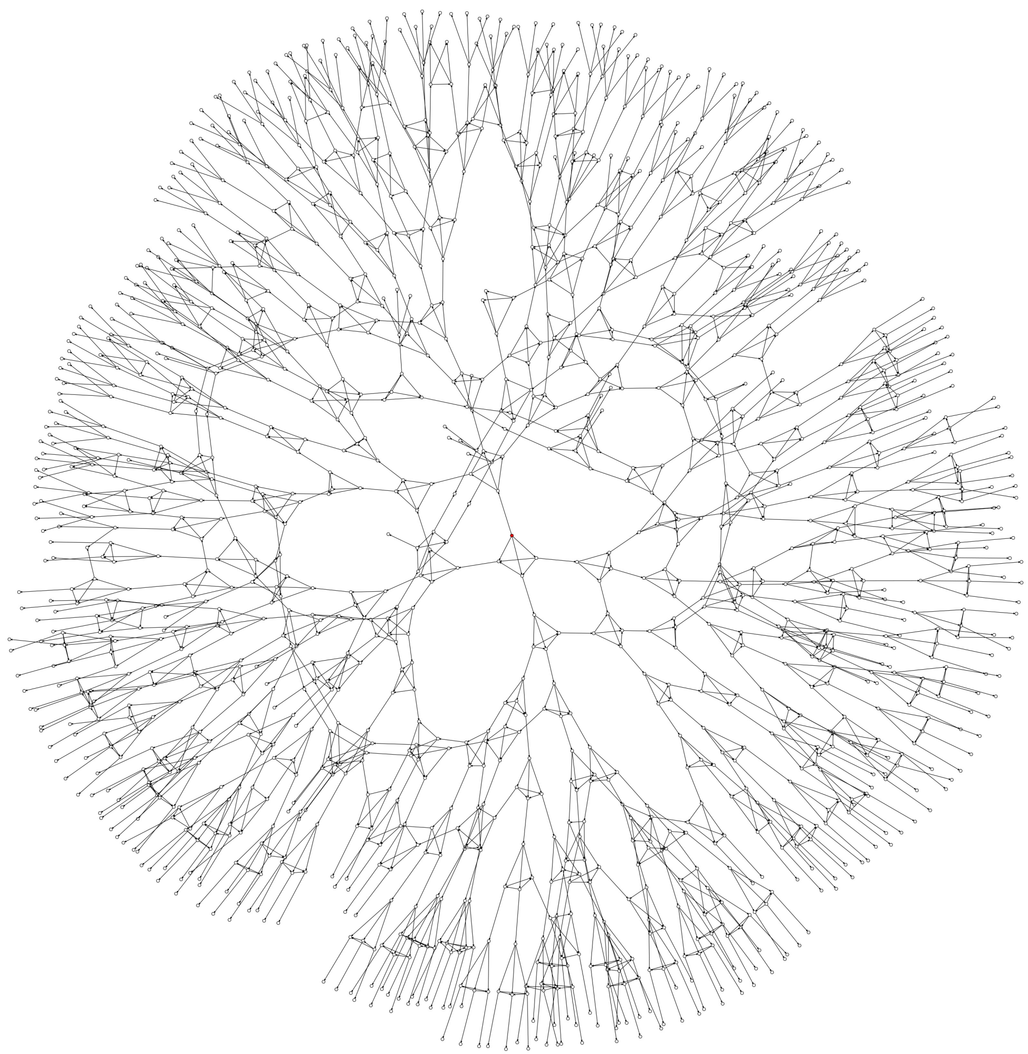
⑥ Spectra of groups of intermediate growth.

$G = \langle a, b, c, d \rangle$ – "first" group of intermediate growth. (act on 

)

$\tilde{G} = \langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \rangle$ – "first" torsion free group of intermediate growth.

$$\tilde{G} \overset{\psi}{\twoheadrightarrow} \tilde{G}, \quad \psi : \begin{cases} \tilde{a} \to a \\ \tilde{b} \to b \\ \tilde{c} \to c \\ \tilde{d} \to d \end{cases}$$



Th. (Artem Dudko, Gri.)

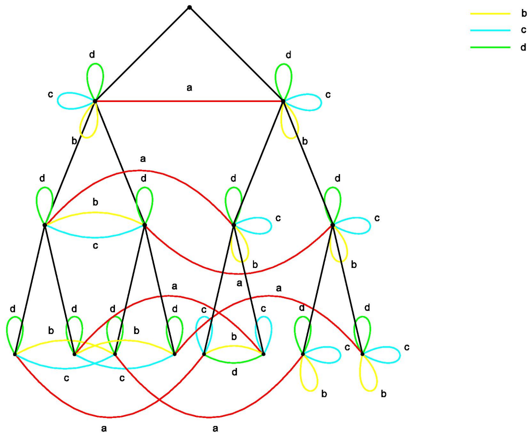
a) $sp(M_{\tilde{G}}) = [-1, 1]$

b) $sp(M_G) = [-\frac{1}{2}, 0] \cup [\frac{1}{2}, 1]$

Spectra of Cayley graphs of $\tilde{G}$, G

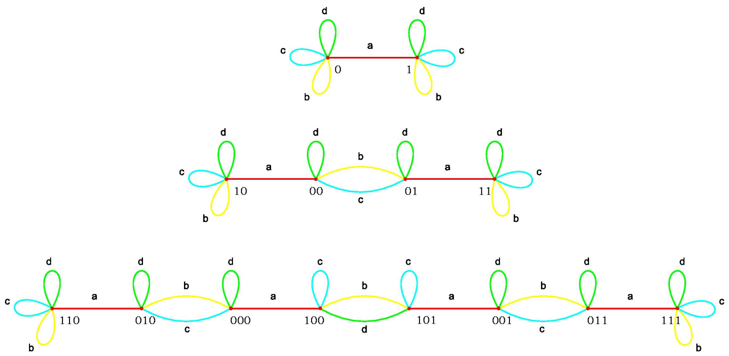
c) (Bartl..., Gr. 2000)

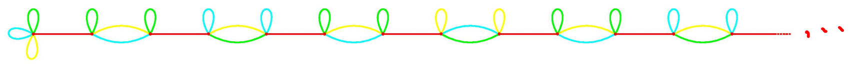
$\forall \xi \in \partial T, \quad sp(\Gamma_\xi) = [-\frac{1}{2}, 0] \cup [\frac{1}{2}, 1]$



Schreier graphs

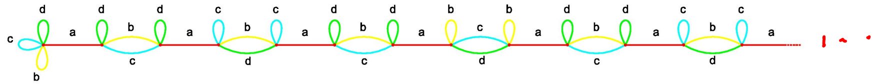
T_n





Schreier graphs Γ_ξ, $\xi \in \partial T$

aperiodic order $\rightarrow$



if $w(b) = w(c) = w(d)$, then $Sp(M_w)$ is union of two intervals.

if $w(b), w(c), w(d)$ are not all equal, then $Sp(M_w)$ is a Cantor set of Lebesgue measure zero.

D. Lenz, T. Nagnibeda, Gri, ... , 2015

Reduction to Random Schroedinger Operator

Q. What is the spectrum of M_G in the case when not all $w(b), w(c), w(d)$ are equal?