Amenability, Group C^*-algebras and Operator spaces (WEP and LLP for $C^*(G)$)

Gilles Pisier
Texas A&M University

Main Source...

E. Kirchberg, On nonsemisplit extensions, tensor products and exactness of group C^*-algebras. Invent. Math. **112** (1993), 449–489.

C^*-algebras

A C^*-**algebra** is a closed self-adjoint subalgebra

$$A \subset B(H)$$

of the space of bounded operators on a Hilbert space H
An **operator space** is a (closed) subspace $E \subset A$ of a C^*-algebra
I will restrict to unital C^*-algebras
The norm on the $*$-algebra A satisfies

$$\forall x, y \in A \quad \|xy\| \leq \|x\|\|y\| \quad \|x\| = \|x^*\| \quad \|x^*x\| = \|x\|^2$$

Such norms are called C^*-**norms**
After completion (or if A is already complete):

there is a unique C^*-norm on A

Then any such A can be written as

$$A = \overline{\mathrm{span}}[\pi(G)]$$

for some discrete group G and some unitary representation

$$\pi : G \to B(H)$$

of G on H
Typical operator space

$$E = \overline{\mathrm{span}}[\pi(S)] \quad S \subset G$$

Throughout I will restrict to discrete groups

Tensor products

On the *algebraic* tensor product (BEFORE completion)

$$A \otimes B$$

there is a minimal and a maximal C^*-norm denoted by

$$\| \ \|_{\min} \quad \text{and} \quad \| \ \|_{\max}$$

and in general

$$\| \ \|_{\min} \leq_{\neq} \| \ \|_{\max}$$

C^*-norms

$$\forall x, y \in A \quad \|xy\| \leq \|x\|\|y\| \quad \|x\| = \|x^*\| \quad \|x^*x\| = \|x\|^2$$

Nuclear pairs

Let A, B be C^*-algebras

Definition

The pair (A, B) is said to be a nuclear pair if the minimal and maximal C^*-norms coincide on the *algebraic* tensor product $A \otimes B$, in other words

$$A \otimes_{\min} B = A \otimes_{\max} B$$

Definition

A C^*-algebra A is called nuclear if this holds for ANY B

Let

$$A_1 = \overline{\mathrm{span}}[\pi(G_1)] \subset B(H_1) \quad A_2 = \overline{\mathrm{span}}[\pi(G_2)] \subset B(H_2)$$

for some unitary rep. $\pi_j : G_j \to B(H_j)$ then

$$A_1 \otimes_{\min} A_2 = \overline{\mathrm{span}}[(\pi_1 \otimes \pi_2)(G_1 \times G_2)] \subset B(H_1 \otimes H_2)$$

Let

$$\Pi = \oplus_p \pi_1.\pi_2 : G_1 \times G_2 \to \mathcal{H} = \oplus_p H_p$$

over all pairs $p = (\pi_1, \pi_2)$ of unitary rep. on the same H_p with commuting ranges, then

$$A_1 \otimes_{\max} A_2 = \overline{\mathrm{span}}[\Pi(G_1 \times G_2)] \subset B(\mathcal{H})$$

Let us denote

$$C^*_\pi(G) = \overline{\mathrm{span}}[\pi(G)] \subset B(H_\pi)$$

The fundamental cases of interest are

$$\pi = \lambda_G$$

left regular rep.
leading to the reduced C^*-algebra of G

$$C^*_\lambda(G) \subset B(\ell_2(G))$$

and

$$\pi = \pi_U$$

universal representation leading to the full C^*-algebra of G

$$C^*(G) \subset B(\mathcal{H})$$

Our main interest will be

$$C^*(G)$$

$$\mathbb{C}[G] \subset C^*(G)$$

$$\forall x = \sum\nolimits_{g \in G} x(g) g \in \mathbb{C}[G]$$

$$\|x\|_{C^*(G)} = \sup\nolimits_{\pi \in \widehat{G}} \|\sum\nolimits_{g \in G} x(g)\pi(g)\|$$

$$\forall x = \sum\nolimits_{g \in G} x(g) \otimes g \in M_k \otimes \mathbb{C}[G] \quad (x(g) \in M_k)$$

$$\|x\|_{M_k(C^*(G))} = \sup\nolimits_{\pi \in \widehat{G}} \|\sum\nolimits_{g \in G} x(g) \otimes \pi(g)\|_{M_k(B(H_\pi))}$$

Let G_1, G_2 discrete groups

$$C^*(G_1) \otimes_{\max} C^*(G_2) \simeq C^*(G_1 \times G_2)$$

$$C^*_\lambda(G_1) \otimes_{\min} C^*_\lambda(G_2) \simeq C^*_\lambda(G_1 \times G_2).$$

$$C^*(G_1) * C^*(G_2) \simeq C^*(G_1 * G_2)$$

$$C^*(G) \subset C^*_\lambda(G) \otimes_{\max} C^*_\lambda(G).$$

diagonal embedding $x \mapsto x \otimes x$

Basic classical fact :

G is **amenable**

$\Leftrightarrow C^*_\lambda(G)$ is nuclear

$\Leftrightarrow C^*(G)$ is nuclear

$\Leftrightarrow C^*_\lambda(G) = C^*(G)$

The fundamental pair

$$\mathcal{B} = B(\ell_2)$$

or more generally $B(H)$ for a general Hilbert space H

$$\mathcal{C} = C^*(\mathbb{F}_\infty)$$

or more generally $C^*(\mathbb{F})$ for a general free group $\mathbb{F}$

Theorem (Kirchberg,1990's)

The pair $(\mathcal{B}, \mathcal{C})$ is a nuclear pair.

Definition

- A is WEP if $(A, \mathcal{C})$ is a nuclear pair.
- A is LLP if $(A, \mathcal{B})$ is a nuclear pair

Fundamental examples : $\mathcal{B}$ is WEP and $\mathcal{C}$ is LLP

Theorem (Kirchberg,1990's)

If A is LLP and B WEP then (A, B) is a nuclear pair.

Definition

A discrete group G is said to be WEP (resp. LLP) if its *full* C^*-algebra $C^*(G)$ is WEP (resp. LLP)

It is known that $WEP \not\Rightarrow LLP$ ([JP,1995])

Main Open problem (Kirchberg's conjecture)

$$LLP \overset{?}{\Rightarrow} WEP$$

Equivalently:

Is any free group WEP ?

This (easily) reduces to $\mathbb{F}_2$

WEP passes to subgroups (but not to C^*-subalgebras) and is determined by countable subgroups

Kirchberg's conjecture is true IFF
the Connes embedding problem has a positive answer,
This would imply the conjecture that
every group G is hyperlinear
A group is "hyperlinear" if the von Neumann algebra M_G generated by λ_G embeds in the ultraproduct of a family of matrix algebras, (note: sofic implies hyperlinear) **More explicitly:**
G is hyperlinear iff $\forall S \subset G$ finite, $\forall \varepsilon > 0\ \exists N < \infty\ \exists \psi :\ G \to U_N$

$$\text{where } U_N = \{N \times N - \textit{unitary matrices}\}$$

such that

$$\forall s, t \in S \quad N^{-1}\mathrm{tr}|\psi(s)\psi(t) - \psi(st)|^2 < \varepsilon,$$

and

$$|N^{-1}\mathrm{tr}(\psi(e)) - 1| < \varepsilon \quad \text{and} \quad \forall t \in S, t \neq e \quad |N^{-1}\mathrm{tr}(\psi(t))| < \varepsilon.$$

Examples ?

$$\{\text{amenable}\} \subset \text{hyperlinear}$$
$$\{\text{ residually finite}\} \subset \text{hyperlinear}$$
$$\text{(Connes, S. Wassermann 1976)}$$

More generally

$$\{\text{residually amenable}\} \subset \text{hyperlinear}$$

or even

$$\{\text{residually hyperlinear}\} \subset \text{hyperlinear}$$

Non residually finite example:
The Baumslag-Solitar group
$BS(3,2) = < a, b \mid ab^3a^{-1} = b^2 >$
is hyperlinear (Radulescu 2008)

Burnside groups ?

Characterization of LLP groups

Let A be a (unital) C^*-algebra
A mapping $f: G \to A$ is called positive definite if
$\forall n, \forall t_1, \cdots, t_n \in G$

$$[f(t_i^{-1} t_j)] \in M_n(A)_+$$

Proposition (Ozawa, 2001)

Let $\mathcal{B}/\mathcal{K}$ denote the Calkin algebra
A group G is LLP iff any unital positive definite

$$f: G \to \mathcal{B}/\mathcal{K}$$

admits a unital positive definite lifting

$$\tilde{f}: G \to \mathcal{B}$$

More generally:

Let $\varphi : G \to \mathbb{C}$ be a positive definite function.
Assume $\forall S \subset G$ finite, $\forall \varepsilon > 0$ $\exists N < \infty$ $\exists \psi : G \to U_N$ such that

$$\forall s, t \in S \quad N^{-1}\mathrm{tr}|\psi(s)\psi(t) - \psi(st)|^2 < \varepsilon,$$

and

$$|\varphi(t) - N^{-1}\mathrm{tr}(\psi(t))| < \varepsilon.$$

If G is LLP then this approximation can be made by restrictions to S of positive definite functions $\psi : G \to M_N$

Application (hyperlinear or sofic case):

$$\varphi(t) = \begin{cases} 1 & \text{if t=1} \\ 0 & \text{otherwise} \end{cases}$$

Hyperlinear + LLP $\Rightarrow$ a **reinforcement** of the hyperlinear approximation

Examples ?

$$\{\text{amenable}\} \cup \{\text{ free groups}\} \subset \{LLP\}$$

[and similarly for WEP if Kirchberg's conjecture is correct...]
It is not easy to produce examples or counterexamples....besides those !

LLP passes to subgroups
and to free products of LLP groups (P. 1996)

Open problem: Is $\mathbb{F}_2 \times \mathbb{F}_2$ LLP ?

Problem Find more examples of groups
either with or without LLP !

Characterization of WEP groups

Let $K : G \times G \to \mathbb{C}$ be a kernel. We still denote by $K : \mathbb{C}[G] \times \mathbb{C}[G] \to \mathbb{C}$ the associated sesquilinear form antilinear (resp. linear) in the first (resp. second) variable. Let

$$\mathbb{C}[G]_+ = \{a \mid \exists x \in \mathbb{C}[G] \;\; a = x^* x\}$$

We will say that K is **bipositive** if there is a mapping $T : \; G \to H$ such that
(i) $K(x, y) = \langle T(x), T(y) \rangle$ (K is positive semi-definite)
(ii) $\begin{cases} K(1,1) = 1 \\ K(a, b) \geq 0 \;\forall a, b \in \mathbb{C}[G]_+ \end{cases}$
Then K defines a state f on $\mathbb{C}[G]$ and hence on $C^*(G)$ simply by

$$f(x) = K(1, x) = \langle T(1), T(x) \rangle$$

K is called **"self-polar"** (cf. Woronowicz, 1973) on G if in addition the state f on $\mathbb{C}[G]$ defined by $f(a) = K(1, a)$ is such that the functionals of the form $x \mapsto K(b, x)$ with $0 \leq b \leq 1$ are pointwise dense in the set of those f' such that $0 \leq f' \leq f$.
Then K extends to a state on $\overline{C^*(G)} \otimes C^*(G)$
Any bipositive kernel is dominated by some selfpolar one
Self-polar kernels are "maximal" in the following sense:
any bipositive kernel K' with $K'(1, x) \leq K(1, x)\ \forall x$
must satisfy $K'(x, x) \leq K(x, x)\ \forall x$.
(cf. Woronowicz, 1973)

Example 1 Let h be Hilbert-Schmidt on H_π with $\mathrm{tr}(h^*h) = 1$.
Then

$$K(x, y) = \mathrm{tr}(\pi(x)^* h^* \pi(y) h)$$

is bipositive.
When $\dim(H_\pi) < \infty$ we call these *matricial* bipositive kernels.

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Example 2 (generalization) Assume $\pi(G) \subset M$ with (M, τ) semifinite von Neumann algebra with trace τ
Let $h \in L_2(\tau)$ be a unit vector. Then

$$K(x, y) = \tau(\pi(x)^* h^* \pi(y) h)$$

is bipositive.

In particular $K(x, y) = \langle x, y \rangle_{\ell_2(G)} \quad \pi = \lambda_G, h = 1$

Theorem

The following are equivalent

- *(i) G is WEP (i.e. $C^*(G)$ is WEP)*
- *(ii) Any self-polar kernel is a pointwise limit of matricial bipositive kernels*
- *(iii) For any bipositive kernel K there is a net of matricial bipositive kernels K_α such that*

$$K \leq \lim_\alpha K_\alpha \text{ on } G \times G.$$

An important result of Taka Ozawa about the special case of $\mathbb{F}_\infty$:
To show $G = \mathbb{F}_\infty$ is WEP (i.e. Kirchberg's conjecture) it suffices to show (iii) above on $S \times S$ where S is the set of generators of $G = \mathbb{F}_\infty$
By Grothendieck's inequality one can prove that

- (iii)' For any bipositive kernel K there is a net of matricial bipositive kernels K_α such that (!)

$$K \leq (4/\pi) \lim_\alpha K_\alpha \text{ on } S \times S$$

This is closely related to an unpublished result due to Haagerup:

Theorem (Haagerup)

A C^*-algebra A is WEP iff

$$\forall n \forall x_1, \cdots, x_n \in A \quad \| \sum \bar{x}_j \otimes x_j \|_{\bar{A} \otimes_{\max} A} = \| \sum \bar{x}_j \otimes x_j \|_{\bar{A} \otimes_{\min} A}$$

When $A = C^*(G)$ and $x_j \in \mathbb{C}[G]$ if $T = \sum \bar{x}_j \otimes x_j$ satisfies $T^* = T$

$$\| \sum \bar{x}_j \otimes x_j \|_{\bar{A} \otimes_{\max} A} = \sup \{ \sum K(x_j, x_j) \mid K \ \textit{bipositive} \}$$

$$\| \sum \bar{x}_j \otimes x_j \|_{\bar{A} \otimes_{\min} A} = \sup \{ \sum K(x_j, x_j) \mid K \ \textit{matricial bipositive} \}$$

About property T

$$\forall x \in \mathbb{C}[G] \quad \|x\|_{\ell_2(G)} = \left(\sum\nolimits_G |x(t)|^2\right)^{1/2}$$

Theorem

Assume either that G is WEP
or that G is LLP and hyperlinear, then $\forall n\ \forall x_1, \cdots, x_n \in \mathbb{C}[G]$

$$(**) \quad \sum \|x_j\|^2_{\ell_2(G)} \leq \|\sum \bar{x}_j \otimes x_j\|_{\overline{C^*(G)} \otimes_{\min} C^*(G)}$$

This (and the next theorem) are a reformulation
Kirchberg's factorization property
(but formally more general)

Theorem (Kirchberg, 1994)

If G has property (T) and satisfies (**) then G is residually finite

Theorem

If G has property (T) and satisfies

$$(**) \quad \sum \|x_j\|^2_{\ell_2(G)} \le \|\sum \bar{x}_j \otimes x_j\|_{\overline{C^*(G)} \otimes_{\min} C^*(G)}$$

then G is residually finite

Proof.

By "Hahn-Banach" $(**)$ implies there are π_i's and a net h_i Hilbert-Schmidt on H_{π_i} with $\mathrm{tr}(h_i h_i^*) = 1$ such that for all $x \in \mathbb{C}[G]$

$$(\dagger) \quad \|x\|^2_{\ell_2(G)} \le \lim_i \mathrm{tr}(\pi_i(x)^* h_i \pi_i(x) h_i^*)$$

$$\Rightarrow \forall g \in G \quad 1 \le \lim_i \mathrm{tr}(\pi_i(g)^* h_i \pi_i(g) h_i^*)$$

$$\Rightarrow \|\pi_i(g)^* h_i \pi_i(g) - h_i\|_{S_2} \to 0 \text{ (almost invariant vector)}$$

By Property (T) we can assume $\pi_i(g)^* h_i \pi_i(g) - h_i = 0$ and hence that $\pi_i(x)^* h_i \pi_i(x) h_i^* = \pi_i(x)^* \pi_i(x) h_i h_i^*$, then (by spectral theory), we can assume that all the π_i's are finite dimensional.
Lastly $(\dagger)$ is actually an equality, so the π_i's separate G ∎

Lastly (†) is actually an equality, so the π_i's separate G
Indeed, (†) is of the form

$$\forall x \in \mathbb{C}[G] \quad K_1(x,x) \leq K_2(x,x)$$

with equality on $G \times G$

This forces $K_1 = K_2$
In matrix language $0 \leq [a^1_{ij}] \leq [a^2_{ij}]$ with equality on the diagonal implies $a^1 = a^2$.
Since the finite dim. representations π_i's separate G,
Residual finiteness follows by Malcev's theorem
(finitely generated linear groups are RF)

See the Bekka-delaHarpe-Valette book
"On Kazhdan's property T"
for much more on property T

Andreas Thom's example

Thom, 2010:
$\exists$ a property (T) group G that is
hyperlinear but *NOT* residually finite

What precedes shows that $(**)$ fails and hence G fails *both* WEP and LLP

Note:

$$\{amenable\} \subset WEP \cap LLP$$

A final open Problem

Let A be a C^*-algebra

$$A \text{ both WEP and LLP} \overset{?}{\Rightarrow} A \text{ nuclear}$$

and if we specialize to $A = C^*(G)$ this becomes

$$G \text{ both WEP and LLP} \overset{?}{\Rightarrow} G \text{ amenable}$$

A positive answer would be a very strong way to answer negatively Kirchberg's conjecture
because his conjecture is equivalent to $\mathbb{F}_2$ (or F_∞) is WEP and we know by his Theorem that it is LLP
Thus it seems more reasonable to reformulate the problem as:
Problem: Does there exist a C^*-algebra A, or a group G, that is both WEP and LLP but NOT amenable.

Partial progress

Proposition

Assume that $A \subset B(H)$ is a C^*-algebra such that

- *(i) The inclusion $A^{**} \subset B(H)^{**}$ admits a projection $P : B(H)^{**} \to A^{**}$ with $\|P\|_{cb} \leq 1$.*
- *(ii) The inclusion $A \otimes_{\min} \mathcal{B} \to B(H) \otimes_{\max} \mathcal{B}$ is of norm 1.*

Then A is both WEP and LLP (and conversely).

Theorem

There is an operator space $A \subset B(H)$ satisfying (i) and (ii) that is not exact (and hence does not embed (completely isometrically) into any nuclear C^*-algebra).

Here $\forall v : E \to F$, $\|v\|_n = \|Id_{M_n} \otimes v : M_n(E) \to M_n(F)\|$.

$$\|v\|_{cb} = \sup \|v\|_n$$

Idea of construction

Theorem

Let $A \subset B(H)$ separable C^*-algebra (or operator space). TFAE

- *(i) A is WEP (i.e. $\exists P : B(H)^{**} \to A^{**}$ with $\|P\|_{cb} \leq 1$).*
- *(i) There is $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ and an increasing sequence of finite dim. subspaces $E_n \subset A$ with $\overline{\cup E_n} = A$ such that $\forall n$ the inclusion $E_n \to E_{n+1}$ admits a factorization as*

$$E_n \xrightarrow{v_n} \ell_\infty \otimes M_n \xrightarrow{w_n} E_{n+1}$$

with $\|v\|_n \|w\|_n < 1 + \varepsilon_n$.

- *(ii) For any $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ and any $X_k \subset X_{k+1} \subset \ldots \subset A$ with $\overline{\cup X_k} = A$ there is a subsequence $E_n = X_{k(n)}$ satisfying (i).*

Recall $\forall v : E \to F$, $\|v\|_n = \|Id_{M_n} \otimes v : M_n(E) \to M_n(F)\|$,

$$\|v\|_{cb} = \sup \|v\|_n$$

We construct E_n inside $\mathcal{C} = C^*(\mathbb{F}_\infty)$ starting from $E_0 =$ the span of the generators that is not "exact" as op. space...

Some References

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Thank you !