TETHERED CURVE COMPLEXES AND HOMOLOGICAL STABILITY OF $Mod(S)$ NOTES FROM THE OCTOBER 2016 MSRI WORKSHOP ON MAPPING CLASS GROUPS AND OUTER AUTOMORPHISM GROUPS

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This is joint work with Allen Hatcher.

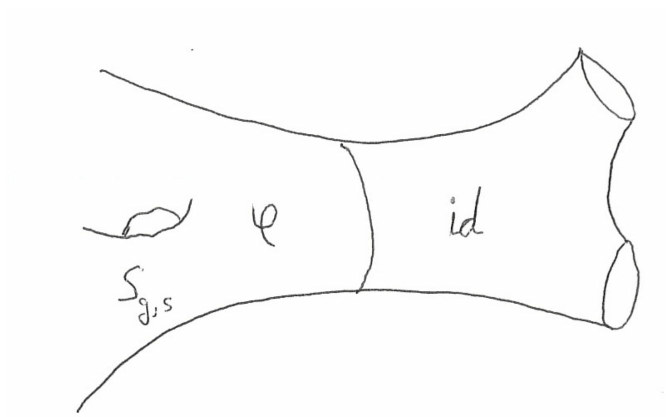
Let $\Gamma_{g,s} = Mod(S_{g,s})$ where $S_{g,s}$ is an orientable surface of genus g with s boundary components. Our mapping classes will fix boundary components.

Question. *What happens as g and s grow?*

There are natural inclusions

$$\mu : \Gamma_{g,s} \hookrightarrow \Gamma_{g,s+1} \qquad \text{if } s \geq 1$$

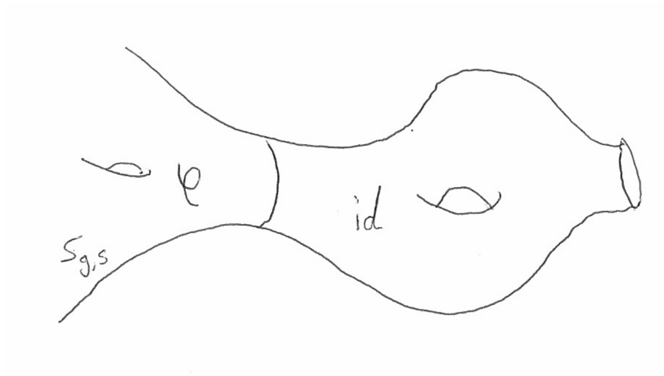
induced by attaching pants and extending a mapping class $\phi \in \Gamma_{g,s}$ as the identity over the pants, as in the diagram



Similarly, attaching a twice-punctured torus along one boundary component induces

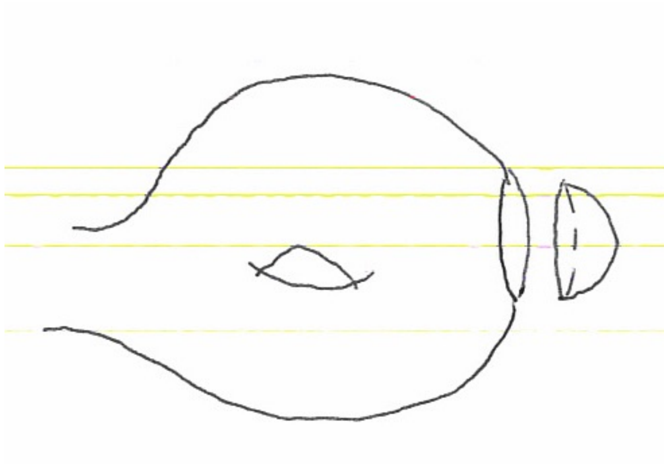
$$\alpha : \Gamma_{g,s} \hookrightarrow \Gamma_{g+1,s} \qquad \text{if } s \geq 1$$

Notes prepared by Edgar A. Bering IV.



We also have

$$\kappa : \Gamma_{g,s} \to \Gamma_{g,s-1} \qquad \text{if } s \geq 1$$



from capping off a boundary component. This is *not* an inclusion.

Theorem (*H.* stability)**.** *The homology group $H_i(\Gamma_{g,s})$ is independent of g and s if $g \gg i$. In fact al of the above maps are isomorphisms on homology.*

The proof is due to Harer. His proof follows the framework put forward by Quillen: Given $G_n \to G_{n+1} \to G_{n+2} \to \cdots$ we would like to find X_n a space (cell complex ideally) with G_n action such that

(1) X_n is highly connected with connectivity growing in n.
(2) X_n/G_n is similarly highly connected.
(3) The stabilizer of a k-simplex is isomorphic to G_{n-k-1}.

In this situation we can then apply the *Equivariant Homology Spectral Sequence*; a major hammer. The inclusions $d_* : H_i(G_{n-1}) \to H_i(G_n)$ occur as a d^1 map in this sequence (the detailed indexing was not included in the talk). The connectivity hypotheses imply that the sequence converges to 0, and an inductive argument shows all other maps are 0. Hence d^1 is an isomorphism. This gives the stability result.

Harer carried out this program using the curve complex of S_g for X_n. This has some problems, the stabilizer of a vertex is not a mapping class group, Dehn twists about the vertex cause trouble. The solution is to consider arc complexes. The proof gets intricate.

Hatcher-V used similar ideas and the sphere complex to show homological stability for $Aut(F_n)$ and $Out(F_n)$. The proof had a gap patched by *Wahl* using an enhanced "tethered" sphere complex.

In this talk we take this idea to $\Gamma_{g,s}$.

Result "Enhanced" curve complexes.

(1) Simpler proof of high-connectivity
(2) Efficient proof of homological stability.

Goals of this talk

(1) Introduce geometric complexes
(2) Sketch connectivity proofs
(3) Show how the Quillen argument plays out

1. Geometric Complexes

Definition. *A* geometric complex *is a simplicial complex defined by:*

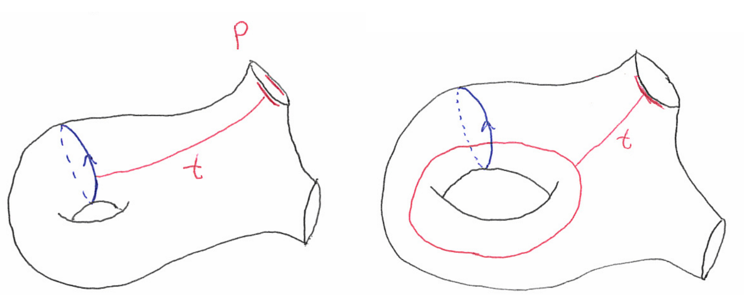
- *Vertices: isotopy classes of some geometric object*
- *k-simplicies: if there are disjoint representatives of the vertices*

For example, the curve complex $\mathcal{C}(S_{g,s})$ is the geometric complex of simple closed curves on $S_{g,s}$.

In this talk we will consider $\mathcal{C}^\circ(S) \subset \mathcal{C}(S)$, the complex of non-separating simple closed curves; and $\mathcal{C}^\circ_\pm(S)$ the complex of *oriented* non-separating simple closed curves. Harer proved that $\mathcal{C}^\circ_\pm(S_{g,s})$ is $(g-2)$-connected, and we will start from this fact.

2. Tethered Curves

We will consider the geometric complex of tethered curves. Fix P a subset of the boundary. The tethered curve complex $TC(S,P)$ is the geometric complex of nonseparating oriented arcs tethered to P via an arc joining a curve to P from the right side of the curve.

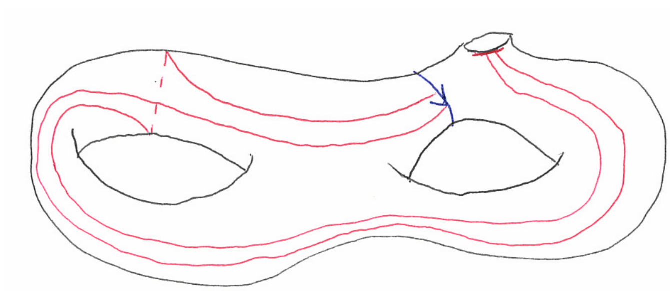


We will also consider doubly tethered curves, joint to P on the right and $Q \subseteq \partial S$ on the left. A chain is a curve tethered to itself, and we may also tether chains, giving the geometric complexes Ch and $TCh(P)$. The choice of sets P and Q ends up not mattering in the proof, so we will supporess them notationally. This gives us a collection of maps

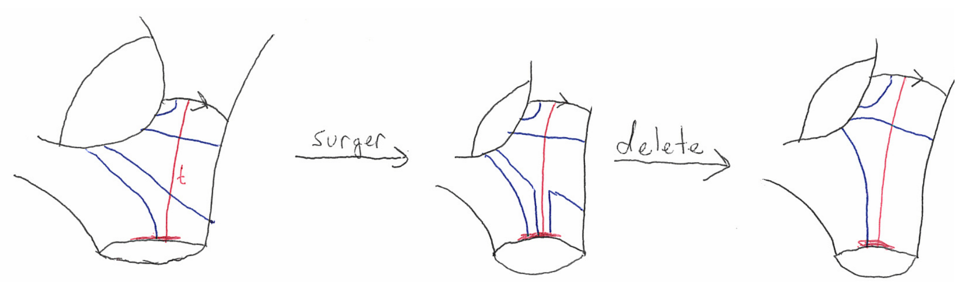
$$\begin{tikzcd} DTC(S) \arrow[r] & TC(S) \arrow[dr] & \\ & & \mathcal{C}^{\circ}_{\pm}(S) \\ TCh(S) \arrow[uu, hook] \arrow[r] & Ch(S) \arrow[ur] & \end{tikzcd}$$

Where every map, except the vertical inclusion, is the forgetful map, dropping the additional tether or chain structure as appropriate.

We want to show all of these complexes are highly connected. All of the arguments are similar, so we will focus on $f : TC \to \mathcal{C}^{\circ}_{\pm}(S)$. The fibers of f are the sets of possible tethers.



We want to show that the fibers are contractible via a surgery argument. Fix a target tether t, for some other tether surger and delete as follows:

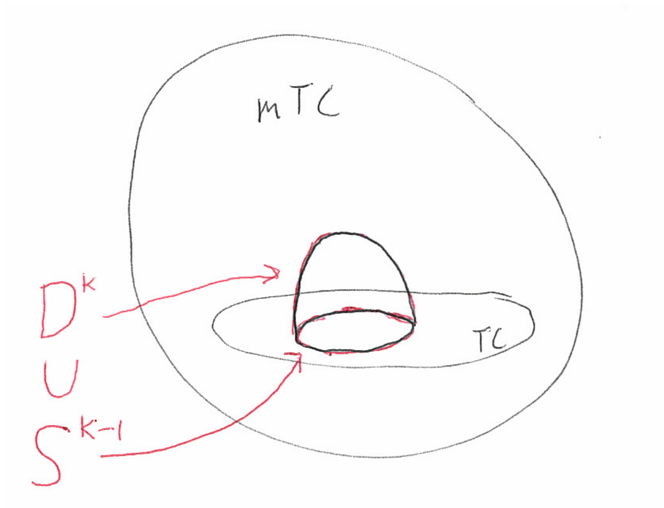


The problem with this approach is that it moves across edges that are not in our complex. Easy solution, allow multiple tethers and get a diagram:

$$\begin{tikzcd} TC \arrow[r, "f"] \arrow[d, hook] & \mathcal{C}^{0}_{\pm} \\ mTC \arrow[ur, "\hat{f}"] & \end{tikzcd}$$

Where mTC is the geometric complex of multiple tethers. The surgery argument shows that the fibers of $\hat{f}$ are contractible, so $\hat{f}$ is a homotopy equivalence.

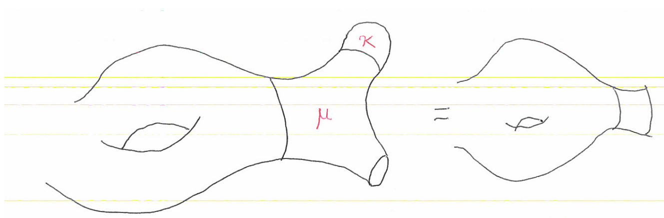
The good news is we can still get information about TC. We have the following situation



The disk D^{k+1} contains "bad" simplices, where all curves have more than two tethers. If there are no bad simplices of any dimension, then D^{k+1} is in TC as desired. There is a link argument that describes how to alter the star of a bad simplex to reduce the dimension of bad simplices in the result. This link argument show that all complexes in the above diagram are $\frac{g-3}{2}$-connected.

Note we cannot use mTC directly in the Quillen argument; stabilizers are not well-behaved.

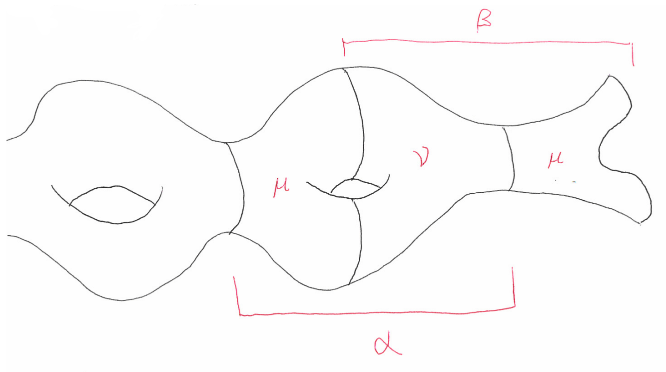
On to homological stability. In the following discussion indexes of homology are suppressed. Using the pants and capping maps, we get the following composition



$$\begin{tikzcd} H_*(\Gamma_{g,s}) \arrow[r, "\mu_*"] \arrow[rr, bend right, "id"'] & H_*(\Gamma_{g,s+1}) \arrow[r, "\kappa_*"] & H_*(\Gamma_{g,s}) \end{tikzcd}$$

This implies μ_* is injective.

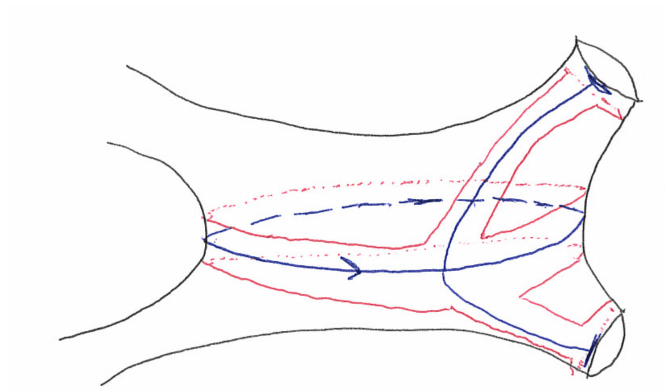
The trick is to decompose α as $\nu \circ \mu$, and analyse $\beta = \mu \circ \nu$.



$$\begin{tikzcd}
\Gamma_{g,s} \arrow[r, "\mu"] \arrow[rr, bend right, "\alpha"'] & \Gamma_{g,s+1} \arrow[r, "\nu"] \arrow[rr, bend left, "\beta"] & \Gamma_{g+1,s} \arrow[r, "\mu"] & \Gamma_{g+1,s+1}
\end{tikzcd}$$

Suppose that β_* is an isomorphism. This implies μ_* is surjective, so μ_* is an isomorphism and therefore also ν_*. Shifting genus, this also shows α_* and κ_* are isomorphisms. This requires $s \geq 1$ and handles all stability there.

So our work is reduced to showing that β_* is an isomorphism in the appropriate degree $g \gg i$. We apply Quillen's method to $DTC(S, P, Q)$ where for simplicity we assume P, Q are in different boundary components. Fix a vertex $v \in DTC$, where $v = (c, s, t)$. If $\phi \in Stab(c, s, t)$ then ϕ has a representative that is identity on a neighborhood of c, s, t and the boundary of S.



The complement of this neighborhood is a surface of genus one less, so we find that $Stab(v) \cong \Gamma_{g-1,s} \hookrightarrow \Gamma_{g,s}$, where the inclusion induces β on homology. Quillen's argument shows β_* is an isomorphism in the relevant degrees, as needed.