COUNTING LATTICE POINTS IN TEICHMULLER SPACE NOTES FROM THE OCTOBER 2016 MSRI WORKSHOP ON MAPPING CLASS GROUPS AND OUTER AUTOMORPHISM GROUPS

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This work is joint with Howard Masur.

Motivation In this talk we are interested in the generecity of pseudo-Anosov elements in the mapping class group of a surface. There are various notions of genericity, today we will focus on the question

Question. *How generic are pseudo-Anosovs?*

The *lattice point counting problem* gives our quantitative basis. Given a lattice, count the number of lattice points in a ball of radius r. For example, for $\mathbb{Z}^2$ acting on $\mathbb{R}^2$

$$|B_R(s) \cap \mathbb{Z}^2 \cdot y| \sim \pi R^2$$

Where we use the notation $A \sim B$ if and only if $\lim \frac{A}{B} = 1$.

Eskin and McMullen give asymptotics for lattices acting on symmetric spaces. We will be talking about $Mod(S)$, so recall

Definition. *The* Teichmüller space S *of a surface of genus* g *with* p *punctures is*

$$\mathcal{T}(S) = \{\textit{marked hyperbolic structures on } S\}/\textit{isotopy}$$

The mapping class group $\Gamma = Mod(S) = Homeo^+(S)$/isotopy acts on $\mathcal{T}(S)$ properly discontinuously, and by isometries with respect to the Teichmüller metric.

Athreya-Bufetov-Eskin-Mirzakhani (ABEM) show that in $\mathcal{T}(S)$ we have

$$|B_R(x) \cap \Gamma \cdot y| \sim \frac{\Lambda(x)\Lambda(y)e^{hR}}{h \cdot vol(\mathcal{T}/\Gamma)}$$

for any $x, y \in \mathcal{T}(S)$, where $h = 6g - 6 + 2p = \dim(\mathcal{T}(S))$ and Λ is the Hubbard-Masur function. It turns out that Λ is constant. The volume is the volume taken in the holonomy measure.

Notes prepared by Edgar A. Bering IV.

Recall that the Nielsen-Thurston classification of Γ partitions Γ into

$$\begin{aligned}\Gamma = &\Gamma_{pA} \text{ pseudo-Anosov elements}\\ &\cup \Gamma_{fo} \text{ finite order elements}\\ &\cup \Gamma_{red} \text{ infinite order elements that fix a 1-manifold}\end{aligned}$$

so we would like to count each of these in pursuit of our question. Mahar shows that

$$\frac{|B_R(x) \cap (\Gamma_{fo} \cup \Gamma_{red}) \cdot y|}{|B_R(x) \cap \Gamma \cdot y|} \to 0$$

which is to say "pseudo-Anosovs are generic". So our question "How generic?" becomes

Question. *How fast is this convergence? (e.g.* $\frac{1}{R}$, $\frac{1}{\log R}$, e^{-R}*)*

This was first asked by Mirzakhani.

Theorem (Dowdall-Masur)**.** *For all* $x, y \in \mathcal{T}(S)$ *there exists a* K *such that*

$$|B_R(x) \cap \Gamma_{fo} \cdot y| \overset{K}{\asymp} e^{\frac{hR}{2}}$$

We use the notation $A \overset{K,C}{\asymp} B$ when $A \leq KB + C$ and $B \leq KA + C$. Further $\overset{K}{\asymp}$ means $\overset{K,0}{\asymp}$.

Most of the remainder of the talk will focus on the proof. Before that, some related work. We are working on

$$|B_R(x) \cap \Gamma_{red} \cdot y| \overset{K}{\asymp} e^{(h-1)R}$$

and asymptotics

$$\frac{|B_R(x) \cap \Gamma_{fo} \cdot y|}{e^{\frac{hR}{2}}} \sim C(x, y)$$

Question. *Count other sets: Dehn twists, multi-twists, conjugacy classes.*

The remainder of the talk is organized along M. Feighn's outline

(1) Statement
(2) Strategy/Goal
(3) Goal is hopeless
(4) Succeed Anyway

1. Strategy of the proof

Observe

- If the statement holds for some x, y then it holds for any x, y
- Γ_{fo} has finitely many conjugacy classes.

The upshot of these observations is that we can choose $\varphi_0 \in \Gamma_{fo}, x = y = x_0 = Fix(\varphi_0)$ and proceede as follows. Suppose $\varphi \in [\varphi_0]$ is such that $\varphi \cdot x_0 \in B_R(x_0)$. Our goal will be to find a nearby ($\leq \frac{R}{2}$) fixed point for φ.

The idea(/exercise) comes form hyperbolic geometry. If φ is a finite order isometry of $\mathbb{H}^2$ the orbit $\varphi^i(x_0)$ has a unique barycenter $b = \varphi(b)$. Slim triangles implies that $d(x_0, b) \leq \frac{1}{2}d(x_0, \varphi x_0) + C$.

Suppose for a moment that we can do this in $\mathcal{T}(S)$. Say b is a thick fixed point such that $d(x_0, b) \leq \frac{R}{2} + C$, so that b is near $f(x_0) \in B_{R/2+C'}(x_0)$. ABEM implies that there are at most $e^{\frac{hR}{2}}e^{C'}$ such $f(x_0)$. Proper discontinuity implies we can get a uniform bound on the number of $\varphi' \in \Gamma$ that coarsely fix $f(x_0)$. This implies there are at most $Ke^{C'}e^{\frac{hR}{2}}$ such $\varphi \in [\varphi_0]$ with $\varphi x_0 \in B_r(x_0)$.

Now for an upper bound choose φ_0 with trivial centralizer. Each $fx_0 \in B_{R/2}(x_0)$ gives us a distinct $\varphi = f\varphi_0 f^{-1}$ such that $d(x_0, \varphi(x_0)) \leq R$. ABEM implies there are at least $e^{hR/2}$ such $f(x_0)$.

1.1. How far away is the fixed set of φ?

Our main tool is the Distance Formula. Let $V \subseteq S$ be an essential subsurface. Define

$$d_V(x, y) = diam_{\mathcal{C}(V)}(\pi_V(\mu_x), \pi_V(\mu_y))$$

unless V is an annulus, in which case take the logarithm. Here μ_x and μ_y are the shortest markings at the points x, y. From Masur-Minsky and Rafi we know that there eists an M, ϵ, K, C such that for all ϵ-thick x, y we have

$$d_{\mathcal{T}}(x, y) \stackrel{K,C}{\asymp} \sum_{V \subseteq S} [d_V(x, y)]_M$$

where

$$[\cdot]_M = \begin{cases} 0, & \cdot < M \\ \cdot, & \cdot \geq M \end{cases}$$

1.2. Applications.

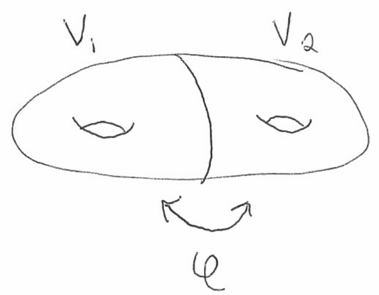
The linear conjugator problem. Masur and Minsky show that there exists a k such that for all $g \in \Gamma_{pA}$, $f \in [g]$ implies $f = wgw^{-1}$ where $|w| \leq k(|f| + |g|)$. J. Tao achieves the same result for $g \in \Gamma_{red} \cup \Gamma_{fo}$. The technique for the finite-order part is to show that for a fixed base marking μ_0 there is a k such that each $\varphi \in \Gamma_{fo}$ has a coarsely fixed marking μ such that

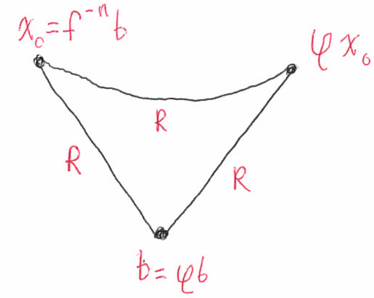
$$d(\mu_0, \mu) \leq kd(\mu_0, \varphi(\mu_0)) + k$$

M. Durham shows there exist a K, C such that for $x \in \mathcal{T}$ and $\varphi \in \Gamma_{fo}$ there exists a fixed point b such that $d_{\mathcal{T}}(x_0, b) \leq Kd_{\mathcal{T}}(x_0, \varphi x_0) + C$. Durham's result gives $|B_R(x) \cap \Gamma_{fo} \cdot y| \leq e^{khR}$, so we need $k = 1/2$.

2. Goal is hopeless

Suppose f is a partial pseudo-Anosov such that $supp(f) = V_1$, and ϕ is the finite order map exchanging V_1 and V_2. Then we have the following situation





To recap

Goal uses	Example
Find fixed point $\leq R/2$ away	Fixed point too far away
ABEM implies at most $e^{hR/2}$ such b	ABEM feels like an over count.

3. Succeed Anyway. Count with more care.

Examine the distance formula. Morally, the lack of control on the number of terms in the summation is the source of error.

Maybe, for any x, y we can find a collection Ω of subsurfaces such that $|\Omega|$ is uniformly bounded, each $V \subseteq S$ with $d_V(x,y) \geq M$, and $\{Z \in \Omega | V \subseteq Z\}$ has a unique $\sup V = Z$. This partitions the sum

$$d_{\mathcal{T}}(x,y) \overset{K,C}{\asymp} \sum_{Z \in \Omega} \Big(\sum_{supV=Z} d_V(x,y) \Big)$$

Terms of this sum look like the distance formula in Z, so we have

$$\asymp \sum_{Z \in \Omega} d_{\mathcal{T}(Z)}(\hat{x}_z, \hat{y}_z)$$

In each Z find $\hat{x}_z, \hat{y}_z$ such that $|d_V(x,y) - d_V(\hat{x}_z, \hat{y}_z)|$ is uniformly bounded. Then

$$\chi(x,y) = \sum_{z \in \Omega} h_Z d_{\mathcal{T}(Z)}(\hat{x}_z, \hat{y}_z) \leq h_S d_{\mathcal{T}}(x,y) + C$$

This new distance formula is the key innovation.

Proposition. *For all x in the thick part*

$$|\{y \in \Gamma \cdot x | \chi(x,y) \leq R\}| \sim e^R$$

Using χ instead of distance we make the strategy work. For $\varphi \in [\varphi_0]$ can find $b = \varphi b$ such that $\chi[x_0, b] \leq \frac{1}{2}\chi(x_0, \varphi x_0) \leq R/2 + C$. The proposition then implies there are at most $e^{hR/2}e^C$ such points, as needed.