FIBRATIONS, SUBSURFACE PROJECTIONS, AND VEERING TRIANGULATIONS NOTES FROM THE OCTOBER 2016 MSRI WORKSHOP ON MAPPING CLASS GROUPS AND OUTER AUTOMORPHISM GROUPS

YAIR MINSKY

This work is Joint with Sam Taylor. An outline

(1) Motivation
(2) Veering Triangulations

1+2=3

1. Subsurface projections and 3-manifolds

Consider a surface bundle over the circle

$$\begin{array}{ccc} S & \longrightarrow & M^3 \\ & & \downarrow \\ & & S^1 \end{array}$$

Let $f : S \to S$ be the monodromy. If it is pseudo Anosov we have stable and unstable foliations λ^+, λ^-. Suppose $Y \subset S$ is an essential subsurface, we get a pairing $d_Y(\lambda^+, \lambda^-)$. To make d_y precise, consider $A(Y)$ the curve and arc complex of Y. Define

$$\pi_Y(\lambda) = [\lambda \cap Y]$$

the finitely many parallel classes of the essential intersections of λ with Y.

$$d_Y(\lambda^+, \lambda^-) = d_{A(Y)}(\pi_Y(\lambda^+), \pi_Y(\lambda^-))$$

The idea from Brock-Canary-Minsky is that short curves correspond to large projections. Precisely

$$\forall s \exists k : d_Y(\lambda^+, \lambda^-) > k \Rightarrow \ell_M(\partial Y) < \epsilon$$

$$\forall k \exists \epsilon : \ell_M(\gamma) < \epsilon \Rightarrow \exists Y : \gamma \subset \partial Y \text{ and } d_Y(\lambda^+, \lambda^-) > k$$

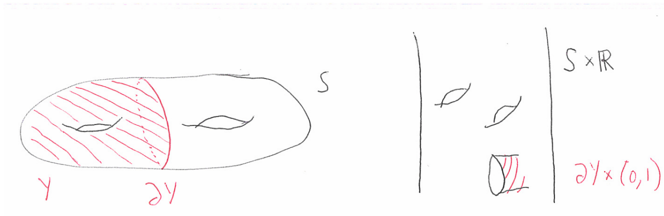
Ugly secrets:

- Quantifiers are non-constructive
- This depends on the choice of fiber S.

Question. *What happens as S changes.*

Picture

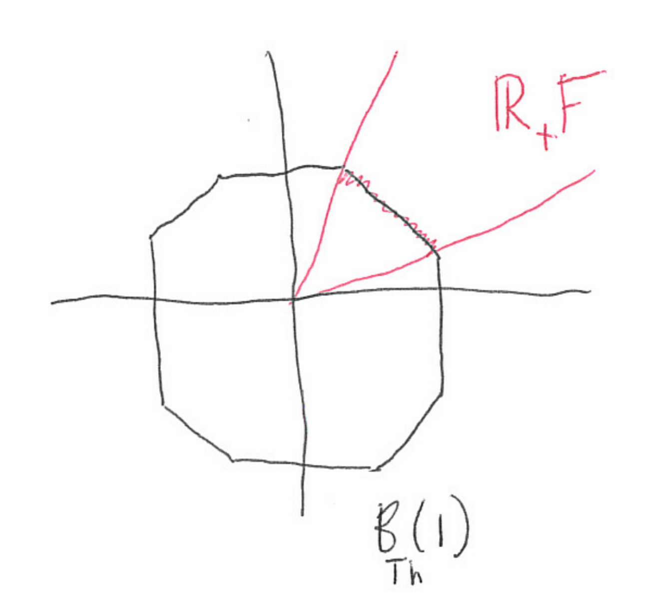
Notes prepared by Edgar A. Bering IV.



In $S \times \mathbb{R}$ we see a tube $\partial Y \times (0,1)$ which maps well into M.

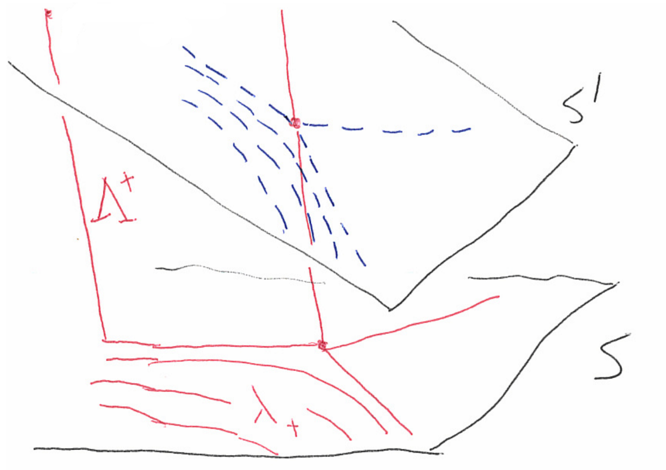
Recall, If $b_1(M) > 1$ then M has infinitely many fibrations.

$$H^1(M) \cong H_2(M, \partial M)$$



There is the Fried-Thurston cone on a face of the unit ball in the Thurston nurm, integral points correspond to different fibrations.

There is also a suspension flow, coming from the vertical flow on $S \times \mathbb{R}$. In a given face F the suspension flow is transverse to all fibers in the different fibrations in the face. The laminations coming from the monodromy can also be suspended into Λ^+, Λ^-, 2-laminations that are transverse to *all* fibers.



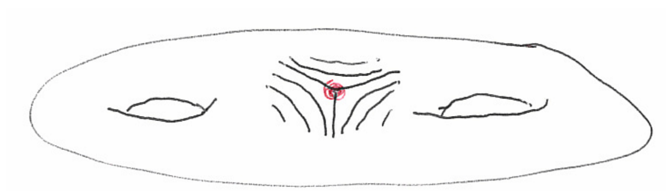
Given such global objects Λ^+, Λ^- we can discuss $d_Y(\Lambda^+, \Lambda^-)$ such that $Y \subseteq S'$ for any fiber $S' \in \mathbb{R}_+ F$.

Question. *Is there an upper bound on $d_Y(\Lambda^+, \Lambda^-)$ independent of the fiber?*

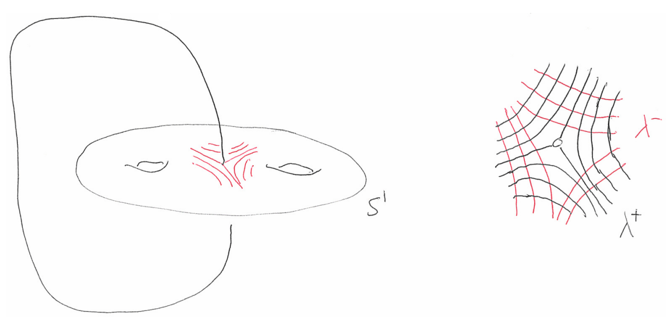
Question. *Fix a fiber F, vary $S \in \mathbb{R}_+ F$. ARe all large projections actually in F?*

2. Veering Triangulations of M

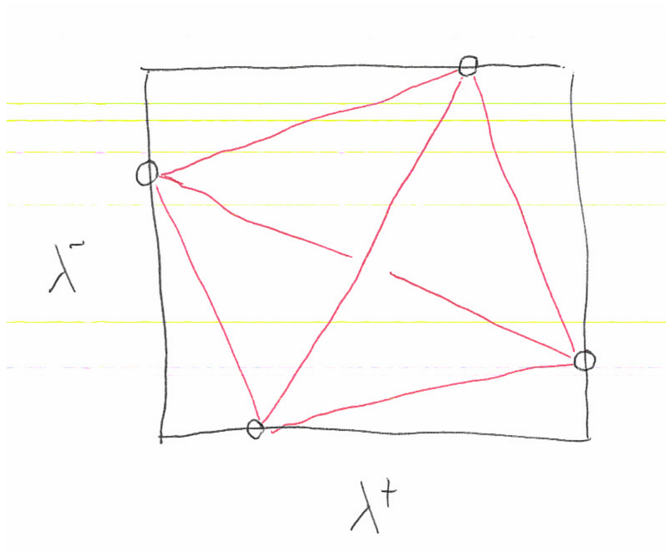
Assume fibers are *fully punctured* ("essential"), that is, the singularities of λ^+, λ^- are at punctures, so M is cusped.



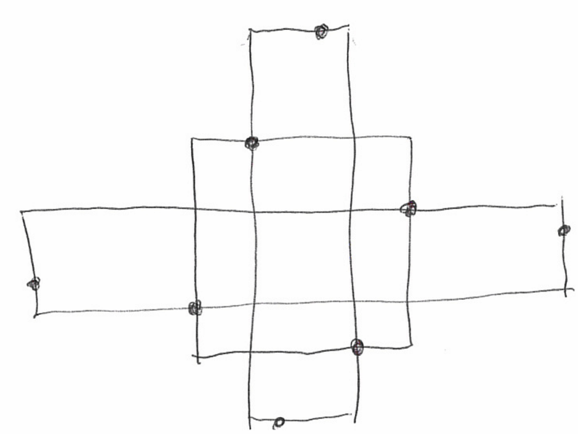
Agol & Guéritaud give a consruction. Fix a fiber S'. Lift to $\tilde{S}$



Find a maximal foliated rectangle $R \subseteq \tilde{S}$. R produces an abstract tetrahedron T_R oriented so that the $+$ edge goes over the $-$ edge



Such things exist, and have one singularity on each edge. This follows from irreducibility. There are in fact infinitely many, non-disjoint such rectangles



Build $X = \cup_R T_R / \sim$ where $\sim$ is "glue according to the picture".

Exercises X is a 3-manifold, $X \to \tilde{S}$ is covering where the fibers are lines.

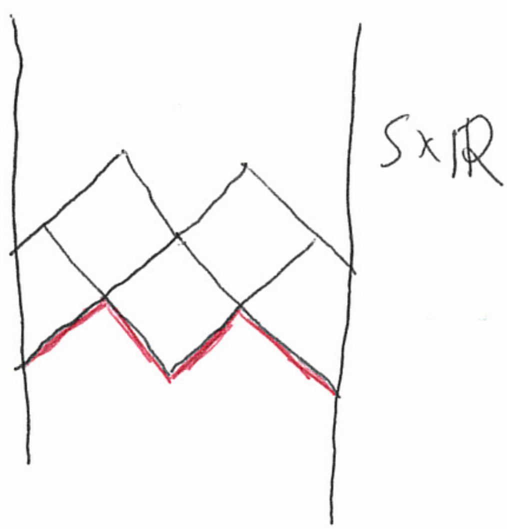
$\tilde{\tau}$ is a triangulation of $\tilde{S} \times R \cong \tilde{M}$. $\pi_1(S)$ and the monodromy acts, both simplicially, giving τ a triangulation of M.

Claim τ depends only on $\mathbb{R}_+ F$ equivalently only on the suspension flow. Consider $\tilde{S} = \tilde{M}/\text{suspension flow} \cong \tilde{S}'$. We have the inclusion $\lambda^{\pm} \hookrightarrow \tilde{\Lambda}^{\pm}/\text{suspension flow}$. Since $\tilde{\Lambda}^{pm}$ do not depend on the fiber we're well-defined.

3. I+II=III

Lemma. *Suppose $Y \subset S \in \mathbb{R}_+ F$ is essential non-annular (this assumption is for simplicity). If $d_Y(\lambda^+, \lambda^-) > 0$ then Y is realized simplicially in a section of τ.*

Definition. *A section of τ is a choice of simpicies in $S \times \mathbb{R}$ like this*



Note that sections are in $S \times \mathbb{R}$. Their image may not be an honest fiber.

Lemma. *If $d_Y(\lambda^+, \lambda^-) > 2$ then Y has a simplicial "pocket" in $S \times \mathbb{R}$.*

U_Y is maximal joining two copies of Y as sections. The pocket U_Y gives two triangulations U^+, U^- of Y and we have $d_Y(\lambda^+, U^+) = 0$ and $d_Y(\lambda^-, U^-) = 0$.

Lemma. *If $d_Y(\lambda^+, \lambda^-) > 8$ then U_Y has a subpocket V_Y that embeds in M and more.*

Consequences of these lemmas.

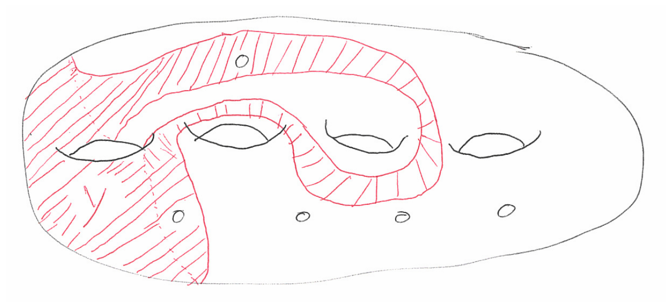
Theorem. *Fix M, fibered face $\mathbb{R}_+F$, τ. For all fibers S, $Y \subset S$*

$$3|\chi(Y)|(d_Y(\lambda^+, \lambda^-) - 8) \leq |\tau|$$

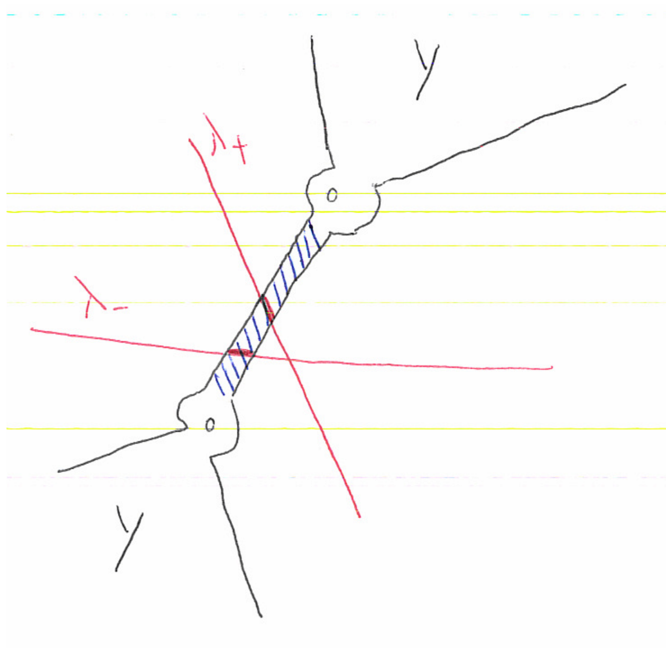
Theorem. *With the same hypotheses. Let F and S be two fibers in $\mathbb{R}_+F$. Suppose $Y \subset S$. Then either*

(1) *Y is isotopic along the flow to a subsurface of F*
(2) *$d_Y(\lambda^+, \lambda^-) \leq 3|\chi(F)| + 8$*

3.1. **Some remarks about the proofs.** Consider a surface with a scary subsurface

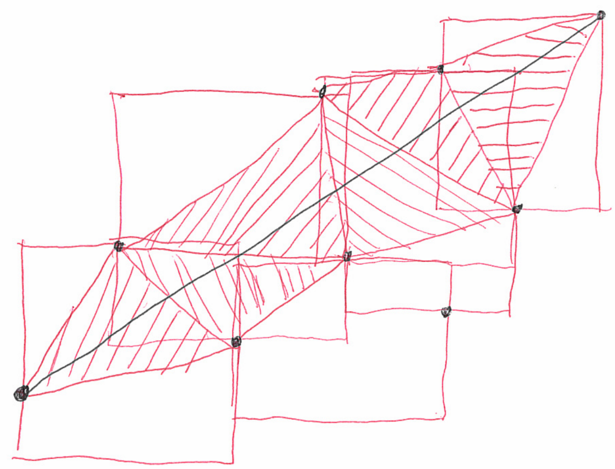


Pull it tight! Problem



These two arcs are isotopic.

So in the first lemma, if Y has positive projection, the tight surface is embedded. Remains to show that Y is a section. In $\tilde{S}$ imagine we need σ in a rectangle to cover a diagonal of Y. Not true.



σ may not be an arc of τ but we can cover σ with a collection of maximal foliated rectangles. Joining singularities gives a τ-hull of σ. Exercise this τ-hull is nice.