CTS AND APPLICATIONS
NOTES FROM THE OCTOBER 2016 MSRI WORKSHOP ON MAPPING CLASS GROUPS AND OUTER AUTOMORPHISM GROUPS

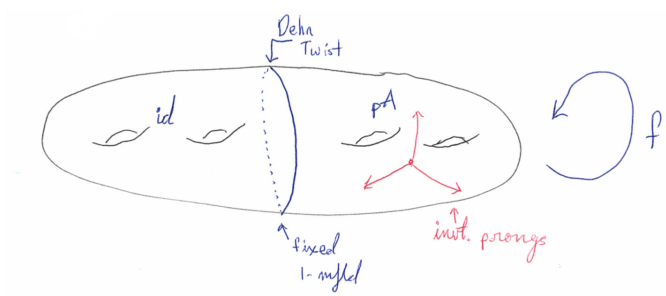
MARK FEIGHN

This is work joint with Michael Handel.

Our setting is $\varphi \in Out(F_n)$. The broad goal is to undertand φ. The specific goal of this talk is to shamelessly advertise CTs as a tool for doing so.

1. Nielsen-Thurston theory, a review

Consider $f \in Mod(S)$ where S is a surface. Thurston normal form: f is in *normal form* if



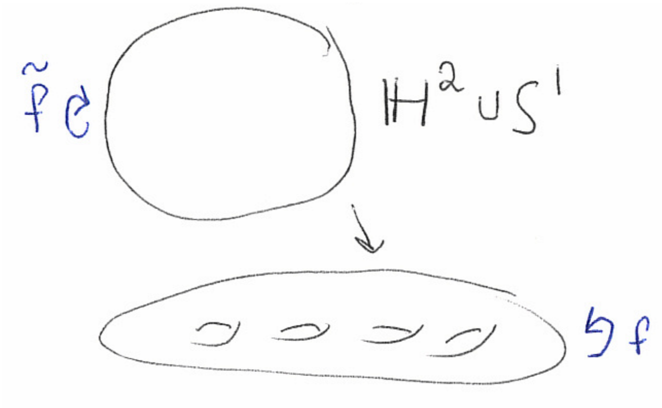
There is a fixed 1 manifold M so that on each component $S \setminus M$ f is either *id* or pseudo Anosov, and f is a multi-twist on M.

Definition. *f is* rotationless *if it can be put into normal form. "No periodic behavior"*

Theorem (Thurston)**.** *There exists a K depending on S such that for all $f \in Mod(S)$, f^K is rotationless.*

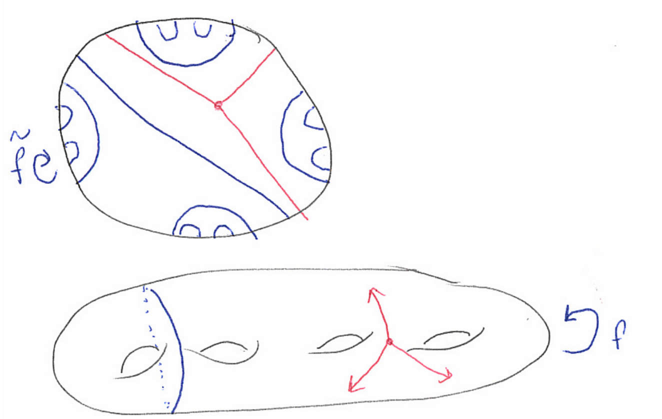
Normal form isn't a perfect invariant, we want something stronger that is an f invariant. Consider the action of a lift $\tilde{f}$ on $\mathbb{H}^2$

Notes prepared by Edgar A. Bering IV.



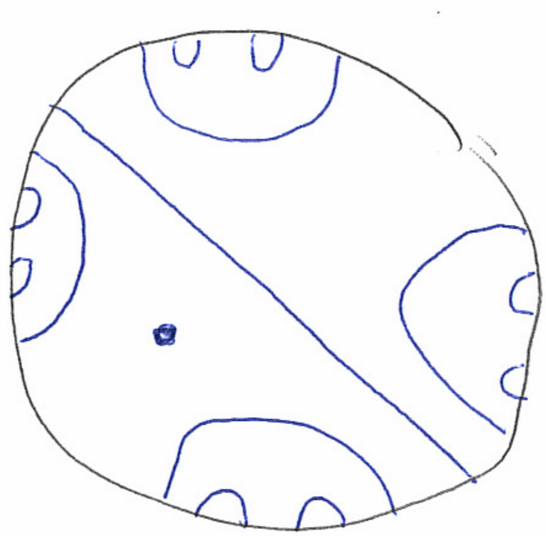
For this lift, it extends to $\partial\mathbb{H}^2 = S^1$. This is an invariant of f, depends on the lift.

Idea Take all extensions of all lifts. Impractical. Finitely many have good info.



Restrict to lifts that fix a given lift of a prong. We see from it 3 neutral fixed points. $|Fix_N(\partial\tilde{f})| = 3$ and if we vary by deck transformations we get the same number.

On an identiy component, pick a point and look at lifts



We see a fixed cantor set $|Fix_N(\partial\tilde{f})| = \infty$.

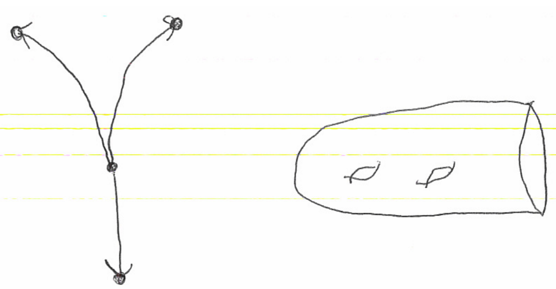
These lifts are interesting.

Definition. *The principal lifts of f are those with at least 3 neutral boundary fixed points.*

Theorem (Thurston)**.** *For rotationless $f \in Mod(S)$*

(1) *There are finitely many families of principal lifts.*
(2) *These fixed sets together with numerical data determine $f \in Mod(S)$.*

This gives a complete invariant.



These indicate the invariant sets at infinity. The Dehn twists are recovered by comparing two lifts on the boundary to get a translation parameter, which is the numerical data.

2. A similar story for $\varphi \in Out(F_n)$

(1) There is an *rotationless* notion
(2) *Principal lift* $\Phi \in Aut(F_n)$ such that $\Phi \in \phi$ and $|Fix_N(\partial\Phi)| \geq 3$ (or 2 with a technical condition).

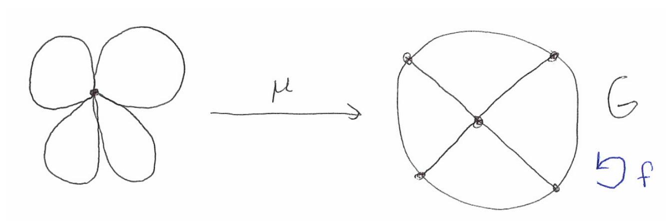
Theorem (Recognition, Feighn-Handel)**.** *For all rotationless $\phi \in Out(F_n)$*

- *Finitely many principal families Φ*
- *$Fix_N(\Phi)$ and numerical data determine φ*
- *$\exists K$ depending on n such that for all $\varphi \in Out(F_n)$, φ^K is rotationless.*

We would like a normal form to make using this complete invariant fun and easy.

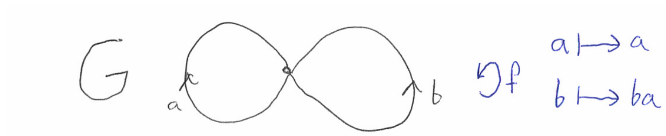
2.1. **Some examples.** Our examples will be topological representatives.

Starting with G a rank n graph with a homotopy equivalence from R_n the n petaled rose called the marking represent φ by $f : G \to G$ a homotopy equivalence taking vertices to vertices and edges to tight edge paths.



As is traditional we will suppress the marking.

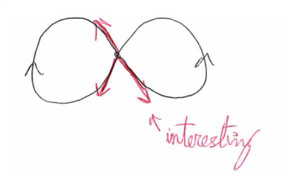
2.1.1. *Example 1.*



Under iteration $b \mapsto ba \mapsto ba^2 \cdots$ so b grows linearly under iteration.

Some features of a topological represnetative to locate

- Fixed edges



- Fixed directions $E \mapsto Eu$
  - Could be linear (u is fixed)
  - Could be superlinear
- Nielsen paths. Some of which are indivisible

$$bab^{-1} \mapsto baa^{-1}b^{-1} = bab^{-1}$$

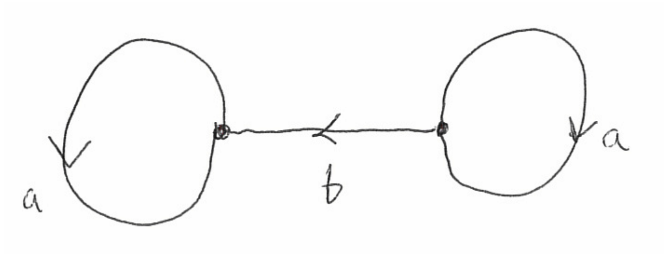
$$ba^i b{-}1$$

The first of these is a basic nielsen paths.

Under iteration, fixed directions give eigenrays.

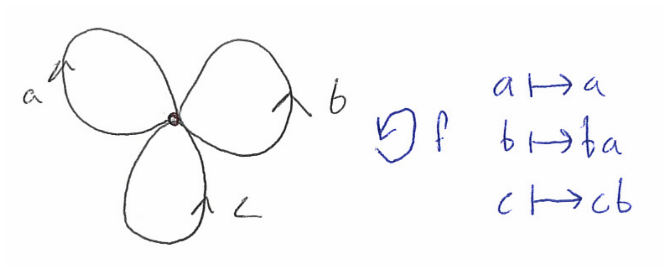
Construction of the invariant.

- Start with fixed vertices and edges
- Add in basic indivisible Nielsen Paths
- Add in superlinear eigenrays

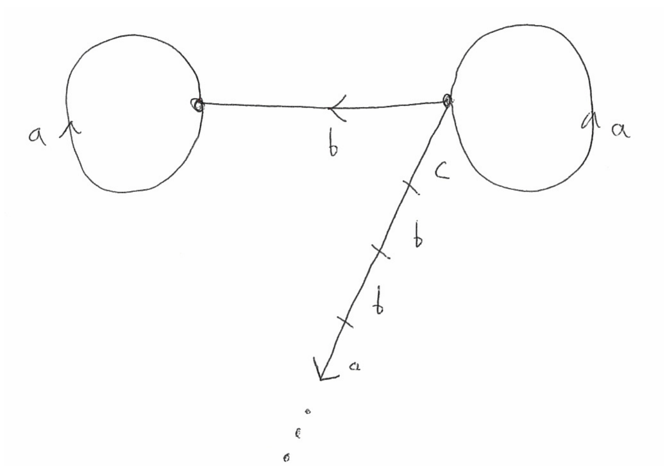


This is an image of $Fix(\Phi)$, providing a graph immersing into G identifying the features.

2.1.2. *Example 2.*



- Fixed edges: $a \mapsto a$
- Fixed directions: $b \mapsto b \cdots$ linear and $c \mapsto c \cdots$ superlinear.
- Same Nielsen paths as before.

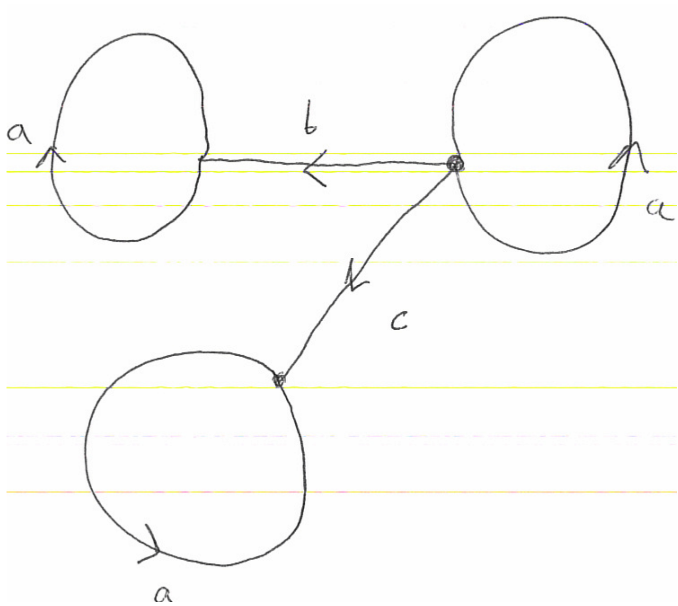


These diagrams are called *Stallings graphs*. The hope is this captures the invariant. These graphs have been studied by Goldstein, Turner, Gersten, and Cooper.

2.1.3. *Example (BAD).* Same graph as before

$$a \mapsto a$$
$$b \mapsto ba$$
$$c \mapsto ca$$

Resulting Stallings graph



Warning $bc^{-1} \mapsto baa^{-1}c^{-1} = bc^{-1}$ is a fixed edge, but not an indivisible Nielsen path, and is not carried by this picture.

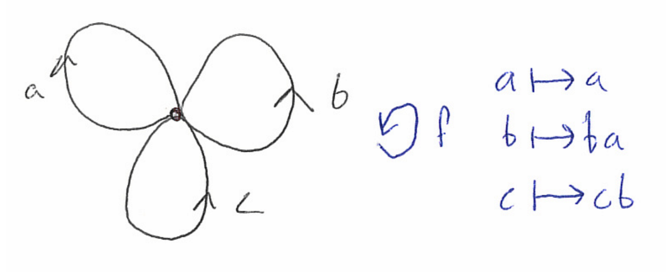
3. CTs

CTs are topological representatives where "what you see is what you get".

3.1. **A brief history of topological representatives.**

- Bestvina-Handel '92
  - Train tracks and Relative Train Tracks
  - Solve the Scott conjecture: $Rank(Fix(\Phi)) \leq n$
- Bestvina-Feighn-Handel
  - Improved relative Train Tracks
  - Tits Alternative for $Out(F_n)$
- CTs Feighn-Handel
  - Recognition Theorem
  - CTs are algorithmic
  - Can construct principal lift fixed data

3.2. **A final example.** Consider the Stallings graph from Example 2.



Define the index of a Stallings graph Γ

$$i(\Gamma) = \sum_{C} [(rk(C) - 1) + \frac{1}{2}|ends(C)|]$$

Here $i(\Gamma) = \frac{3}{2}$. This is $i(\varphi)$ of Gabouriau-Jagon-Levitt-Lustig. GJLL show $i(\varphi) \leq n - 1$.

CTs give another proof of this from their hierarchical structure and also tighter index estimates.