FINDING GEODESICS IN THE CURVE COMPLEX NOTES FROM THE OCTOBER 2016 MSRI WORKSHOP ON MAPPING CLASS GROUPS AND OUTER AUTOMORPHISM GROUPS

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This work is joint with Richard Webb.
Let S be a surface.

Definition. *The* curve complex $\mathcal{C}(S)$ *of* S *is a simplicial complex whose vertices are isotopy classes of simple closed curves, and* k *curves span a simplex when they can be realized disjointly.*

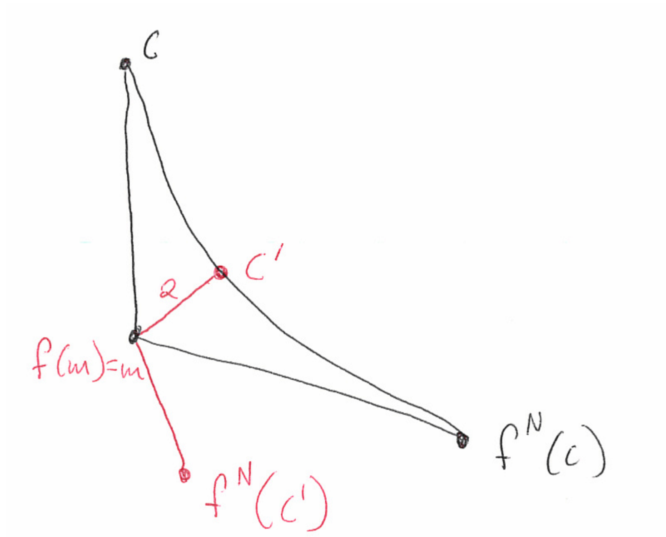
For sufficiently complex surfaces, $\mathcal{C}(S)$ is connected. This is an exercise with surgery.

Theorem (Bell-Webb)**.** *There exists an algorithm to compute a geodesic* $[a, b] \subset \mathcal{C}(S)$ *and it is* $poly(d(a, b))$ *time. (Precisely* $poly(\log i(a, \tau)+\log i(b, \tau))$ *where* τ *is a fixed ideal triangulation.)*

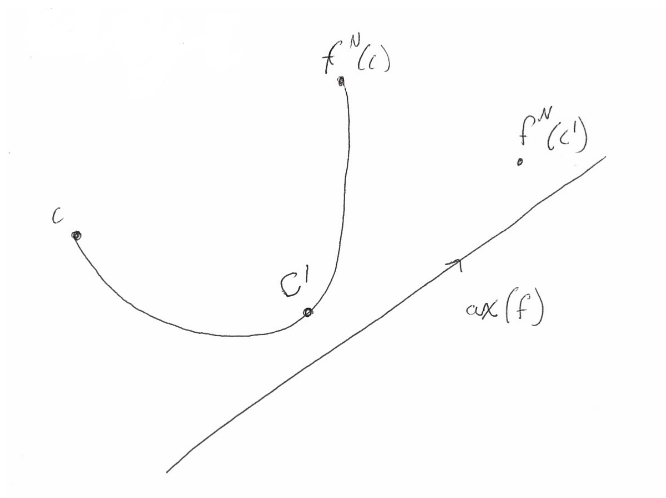
Theorem. *There is a polynomial time algorithm to determine the Nielsen-Thurston type of a mapping class.*

Proof. Proof of theorem 2 from theorem 1 Consider $Mod(S)$ acting on $\mathcal{C}(S)$. If $\varphi \in Mod(S)$ is reducible there is a fixed multicurve m. Start with some curve c. Calculate the geodesic from c to $f^N(c)$ and let c' be the midpoint. By δ-hyperbolicity and the Bounded Geodesic Image theorem the midpoint c' is within 2 of m, so that $d(c', f^N(c')) \leq 4$.

Notes prepared by Edgar A. Bering IV.



In the case f is pseudo Anosov we have the following picture



and conclude that $d(c', f^N(c')) \gg 0$. □

1. Some history of results like Theorem 1

$\mathcal{C}(S)$ is locally infinite, which makes it challenging to approach algorithmically.

We will restrict our attention to adress this problem.

Definition (Masur-Minsky). *A geodesic $a_0, \ldots, a_n \in \mathcal{C}(S)$ is tight if*

$$a_i = \partial N(a_{i-1} \cup a_{i+1})$$

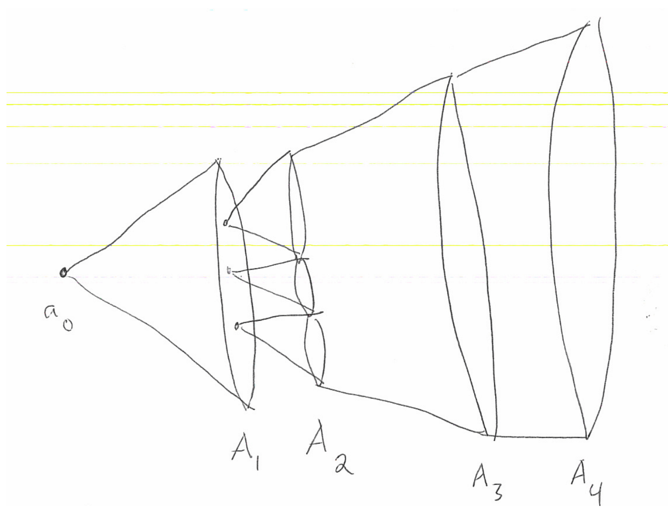
and a_i, a_j fill S when $|i - j| \geq 3$.

Tight geodesics always exist, and by focusing on them we get a handle on the locally infinte nature of $\mathcal{C}(S)$.

Theorem. *There are finitely many tight geodesics* $[a, b]$.

Theorem (Leasure, Shackelton, Watanabe, Webb)**.** *If* $a_0, \ldots, a_n$ *is tight then* $i(a_1, a_n) \leq 2^{|\chi(S)|n} \cot i(a_0, a_n)$.

This bound allows us to search for a_1. There are finitely many possibilities, $a_1 \in A_1$, and A_1 is computable. For each point in a_1 we can repeat this.



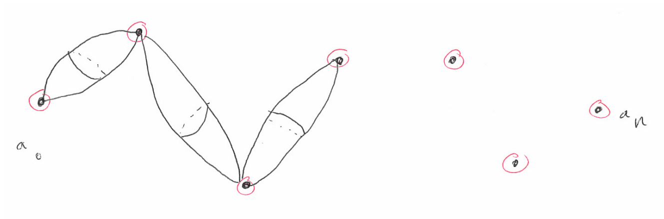
All tight geodesics from a_0 to a_n must pass through these sets, so the problem is computable.

However, this naive approach is a priori searching an exponential graph. There are some optimization tricks to reduce the exponent but the problem is still exponential.

The idea is to pick a better guide through the A sets to avoid checking every tree branch. We would like a set $U \subseteq \mathcal{C}(S)$ with the properties that $a_0, a_n \in U$ and U is quasi-convex, polynomially sized.

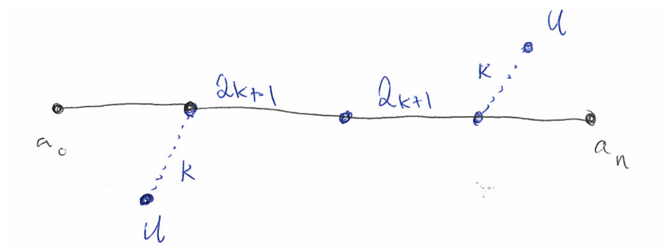
How can we produce such a thing? The train-track splittings of Masur and Minsky. Take a splitting sequence $\tau_0 \to \cdots \to \tau_n$ where τ_0 has a_0 as a vertex cycle and τ_n has a_n as a vertex cycle. Let U be the set of the vertex cycles of all τ_i. Masur and Minsky show this is quasiconvex. Moreover, the work of Agol-Hass-Thurston shows that $|U| \leq poly(n)$.

Computing a tight geodesic with U. Use the A-tree construction to connect the points of U. These trees are all of depth $L = 6K + 2$ where K is the quasi-convexity constant of U, so of size $|A_1|^L$.



The result is a graph with $|A_1|^L \cdot |U| \sim poly(n)$ vertices. Path finding in such a graph is polynomial time.

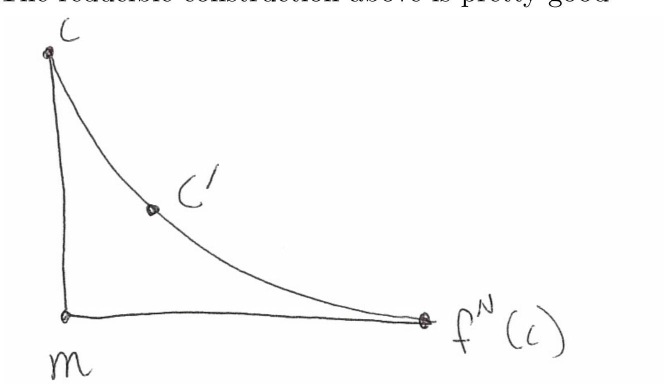
Further, we claim there is a tight geodesic inside this graph.



We can find tight geodesics joining points of U and use quasi-convexity to stitch them together on the overlap.

Other applications of the algorithm.

- Can find an invariant curve system $m \subseteq \sigma(f)$ of the canonical fixed curve. The reducible construction above is pretty good



- But one can show that $m \subseteq \partial N(v \cup f^N(v))$ for some $v \in [c, f^N(c)]$.
- This can also be used to compute the asymptotic translation lengths of pseudo-Anosovs in $\mathcal{C}(S)$.