HYPERBOLIC ACTIONS AND 2^{nd} BOUNDED COHOMOLOGY OF SUBGROUPS OF $Out(F_n)$ NOTES FROM THE OCTOBER 2016 MSRI WORKSHOP ON MAPPING CLASS GROUPS AND OUTER AUTOMORPHISM GROUPS

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This is Joint work with Lee Mosher

Theorem. *Let H be a finitely generated subgroup of $Out(F_n)$. If H is not virtually abelian, then $H^2_b(H,\mathbb{R})$, the second bounded cohomology of H, is infinitely generated.*

1. History

A prior result of Bestvina and Fujiwara gives the same theorem for $Mod(S)$.

Bestvina and Feighn handle the case that H contains an irreducible element.

The Bestvina-Fujiwara argument guides and motivates.

Strategy Find a finite index normal subgroup $N \leq H$ and an action of N on a hyperbolic space with sufficiently many WPD or WWPD elements.

We will see what these words mean in due course.

2. Background

Definition. *$h : \Gamma \to \mathbb{R}$ is a* quasi-morphism *if $|h(\gamma_1\gamma_2) - h(\gamma_1) - h(\gamma_2)|$ is uniformly bounded.*

Examples Homomorphisms, constant functions, and bounded functions. Lifts of a homeomorphism $h : S^1 \to S^1$ to $\mathbb{R} \to \mathbb{R}$—periodicity implies $\tilde{h}$ is a qm.

Definition.

$\widetilde{QH} =$ *vect. space of quasi-morphisms/ subspace of homomorphisms and bounded functions*

There is a sequence

$$1 \to \widetilde{QH}(\Gamma) \to H^2_b(\Gamma,\mathbb{R}) \to H^2(\Gamma,\mathbb{R})$$

So it suffices to show $\widetilde{QH}(\Gamma)$ is infinitely generated.

Notes prepared by Edgar A. Bering IV.

3. Finding quasi-morphisms

Recall. Suppose $\gamma \in \Gamma$ acts loxodromically in a Γ action on a hyperbolic space X. Then $\partial_\pm\gamma$ are fixed points of γ in ∂X.

Definition. *γ is* WWPD *with respect to the action of Γ on X if $\partial_\pm\gamma$ is discrete in $\partial X \times \partial X \setminus \Delta$. If in addition $Stab(\partial_\pm\gamma)$ is virtually abelian then γ is* WPD*.*

WWPD is defined in Bestvina-Bromberg-Fujiwara. WPD in Bestvina-Fujiwara.

Sufficient conditions for $\widetilde{QH}(\Gamma)$ to be infinitely generated

- Γ has a nonelementary action on a δ-hyperbolic X with at least one WPD element.
- There exists a finite index normal subgroup $N < \Gamma$, and N has a nonelementary action on a δ-hyperbolic X with "sufficiently many" WWPD elements.

These conditions are from Bestvina-Fujiwara, though the second is reworked in Handel-Mosher.

- There exists a finite index normal subgroup $N < \Gamma$ and $N \twoheadrightarrow Q < Isom(X)$ where some element of Q is WPD and X is δ hyperbolic.

This condition uses work of Osin to construct an acylindrical action of Q on a different hyperbolic space.

4. Proof of the main theorem

We will apply the "sufficiently many WWPD" elements criterion to H. The proof is by cases.

Recall $\mathcal{L}(\varphi)$ is the set of laminations of $\varphi \in Out(F_n)$. Given H consider $\mathcal{L}(H) = \bigcup_{\varphi\in H} \mathcal{L}(\varphi)$. Our cases will be

- $|\mathcal{L}(H)|$ is finite
- $|\mathcal{L}(H)|$ is infinite

4.1. The infinite-lamination case.

Use as X the free splitting complex. The loxodromic elements are those φ that have some lamination that fills F_n.

Example. $F_4 = \langle A, B, C, D\rangle$, φ is defined by

$$\begin{array}{lll} A \mapsto BA & & C \mapsto DAC \\ B \mapsto BBA & & D \mapsto DBDC \end{array}$$

This is reducible but has a filling lamination.

Theorem. *With $n \geq 3$, $\varphi \in H$, and $\mathcal{F}$ a maximal proper H-invariant free factor system. Suppose φ satisfies*

- *φ is irreducible rel $\mathcal{F}$, that is there is no free factor system $\mathcal{F}'$ such that $\mathcal{F} \sqsubset \mathcal{F}' \sqsubset F_n$ and $\varphi(\mathcal{F}') = \mathcal{F}'$.*
- *$\mathcal{L}(\varphi)$ contains a filling lamination.*

- *$\varphi|_{[A]}$ is trivial for each $A \in \mathcal{F}$.*

Then φ is WWPD on X the free-splitting complex.

Example. φ in F_4 defined by

$$\begin{array}{llll} A & \mapsto A & C & \mapsto \quad Dw_1C \\ B & \mapsto B & D & \mapsto \quad Dw_2Dw_3C \end{array}$$

where $w_1, w_2, w_3 \in \langle A, B, C, D\rangle$ are sufficiently complicated words such that $\overline{\varphi^\infty(C)} = \Lambda_\varphi$ fills. The maximal invariant system is $\mathcal{F} = \{[\langle A, B\rangle]\}$, φ is irreducible relative to $\mathcal{F}$, and $\varphi|_{[\langle A,B\rangle]} = id$. Hence φ is WWPD on X.

Features of the proof. There are no currents, this is all topology. This theorem is then used on the infinitely many laminations of $\mathcal{L}(H)$ to find sufficiently many WWPDs on X for H.

4.2. **The finite lamination case.** This case further splits into $\mathcal{L}(H) = \emptyset$ and $\mathcal{L}(H)$ non-empty. In the interests of time we give an illustrative example of the $|\mathcal{L}(H)| = 0$ case, where H is Kolchin-type.

Again working in F_4 consider the family of automorphisms $\varphi_{u,j}$ defined by

$$\begin{array}{cccccc} A & \mapsto & A & C & \mapsto & C \\ B & \mapsto & BA^j & D & \mapsto & Du \end{array}$$

Where $u \in \langle A, B\rangle$ and $j \in \mathbb{Z}$.

We will consider $H = \langle \varphi_{u,j}\rangle$ for a particular choice of u (not specified).

The idea is to choose an H invariant $F < F_4$, $N(F) = F$. Consider the diagram

$$\begin{array}{ccc} H & \xrightarrow{restrict} & \Gamma < Out(F) \\ & \searrow & \uparrow \\ & & \hat{\Gamma} < Aut(F) \end{array}$$

In our example we first try $F = \langle A, B, C\rangle$. The map takes $\varphi_{u,j} \mapsto i_u\psi^j$ where $\psi(A) = A, \psi(C) = C, \psi(B) = BA$ and i_u is the inner automorphism given by u. This gives $H \to \hat{\Gamma} < Aut(F)$. The group $\hat{\Gamma}$ acts on a simplicial tree.

In this example the resulting action is not useful, but the strategy (that works) is that there is some $F < F_4$ so that the action of $\hat{\Gamma} < Aut(F)$ on some simplicial tree has sufficiently many WWPD elements.