THE GEOMETRY OF HYPERBOLIC FREE GROUP EXTENSIONS
NOTES FROM THE OCTOBER 2016 MSRI WORKSHOP ON MAPPING CLASS GROUPS AND OUTER AUTOMORPHISM GROUPS

SAM TAYLOR

Analogous to the story of hyperbolic surface group extensions we will be studying hyperbolic free group extensions for F_n, $n \geq 3$.

We have

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_n & \longrightarrow & Aut(F_n) & \xrightarrow{\rho} & Out(F_n) & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & F_n & \longrightarrow & E_\Gamma = \rho^{-1}(\Gamma) & \longrightarrow & \Gamma & \longrightarrow & 1 \end{array}$$

analogous to the Birman exact sequence.

Definition. *An automorphism $f \in Out(F_n)$ is* atoroidal *if*

$$\forall \alpha : f^K(\alpha) = \alpha \Rightarrow K = 0$$

where α ranges over conjugacy classes.

Definition. *$\mathcal{F}_n$ denotes the* free factor graph *of F_n. This graph has as vertices conjugacy classes of free factors, with an edge (A, B) if $A \leq B$ or $B \leq A$.*

$Out(F_n)$ acts on $\mathcal{F}_n$.

Theorem (Dowdall-Taylor)**.** *Suppose $\Gamma^{fg} \leq Out(F_n)$. If Γ is purely atoroidal and the orbit map $\Gamma \to \mathcal{F}_n$ is a quasi-isometric embedding then E_Γ is hyperbolic.*

Question. *What do these hyperbolic groups look like?*

Informally and in brief they are "like" hyperbolic surface extensions.

1. Remarks on the Theorem

1.1. The converse is NOT true.

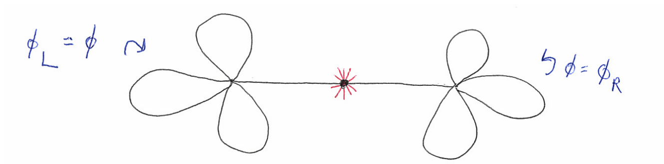
Definition. $\phi \in Out(F_n)$ *is* fully irreducible *(iwip) if*

$$\forall [A] \in \mathcal{F}_\backslash : \phi^k([A]) = [A] \Rightarrow k = 0$$

Notes prepared by Edgar A. Bering IV.

Bestvina-Feighn show that ϕ is iwip if and only if ϕ acts loxodromically on $\mathcal{F}_n$.

Let $\phi \in Out(F_3)$ be an atoroidal iwip, say $a \mapsto abc, b \mapsto bab, c \mapsto cabc$. Take two graph representatives of ϕ on a rose, call them ϕ_L and ϕ_R on the left and right roses of the figure



Define $\Phi = \phi_L \phi_R$. The group

$$E_{\langle\Phi\rangle} = F_6 \rtimes_\Phi \mathbb{Z} = (F_3 \rtimes_\phi \mathbb{Z}) *_\mathbb{Z} (F_3 \rtimes_\phi \mathbb{Z})$$

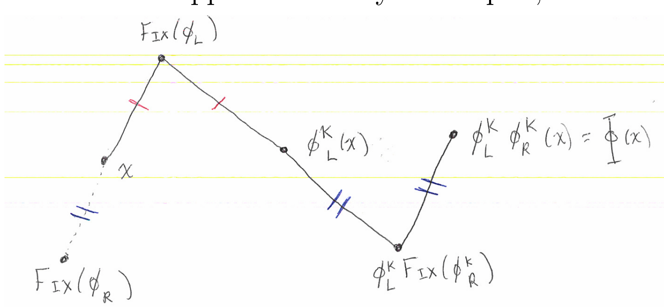
The mapping torus has a bicollared neighborhood of the lift of x, which gives the cyclic splitting. By the Bestvina-Feighn combination theorem $E_{\langle\Phi\rangle}$ is hyperbolic.

Theorem (Brinkman)**.** *$F_n \rtimes_\phi \mathbb{Z}$ is hyperbolic if and only if ϕ is atoroidal.*

1.2. A perfect $Out(F_n)$ graph.

Question. *Does there exist an $Out(F_n)$ graph X^{hyp} that is hyperbolic such that ϕ is X-loxodromic if and only if ϕ is atoroidal.*

The answer is no. Again returning to our example. Suppose we had such a space X^{hyp}. Since $\Phi = \phi_L\phi_R = \phi_R\phi_L$ and ϕ_L, ϕ_R not atoroidal, ϕ_L, ϕ_R are not X loxodromic. Suppose that they are elliptic,



The distance $d(x, \Phi^k(x))$ does not grow with k, so Φ is not loxodromic. The parabolic case is an exercise.

1.3. Toroidal iwips.

There are iwip $\phi \in Out(F_n)$ that are not atoroidal.

Bestvina-Handel show that in this case there exists a punctured surface S with $\pi_1(S) = F_n$ and $f : S \to S$ pseudo Anosov such that $f_* = \phi$.

2. The cosurface graph

$\mathcal{C}^{nonsep}(S) \hookrightarrow \mathcal{F}_n$ for each S with $\pi_1(S) = F_n$ has infinite diameter, each non-separating curve gives a free factor.

Definition. *The* intersection graph *studied by Kapovich-Lustig and Mann-Reynolds.*

Is the same as

Definition. *The* co-surface graph *of* F_n*,* $c\mathcal{S}_n$ *is the graph with vertices primitive conjugacy classes, and an edge* (α, β) *if there exists* S *once punctured with* $\pi_1(S) = F_n$ *such that both* α *and* β *are simple closed curves on* S*.*

Facts about $c\mathcal{S}_n$

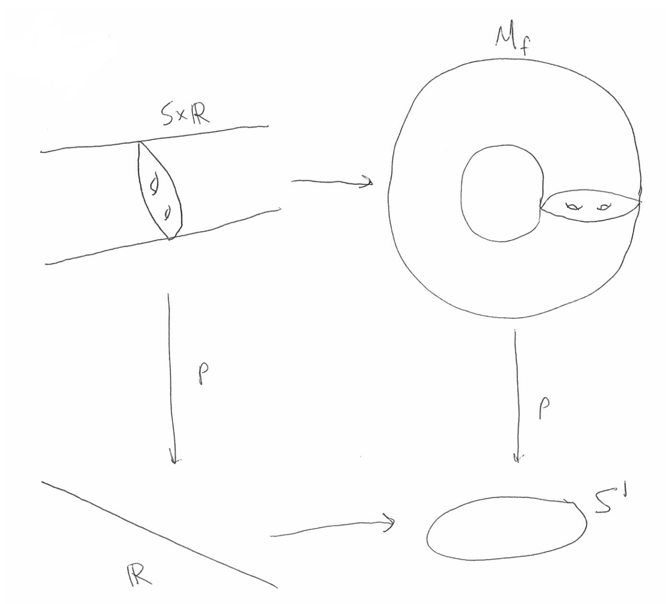
- There is a Lipschitz map $\mathcal{F}_n \to c\mathcal{S}_n$ $(A \mapsto \alpha \in A)$
- It is infinite diameter (Kapovich and Lustig)
- Hyperbolic (Mann and Reynolds)
- $\partial c\mathcal{S}_n \subseteq \partial \mathcal{F}_n$ (Dowdall and Taylor)
- ϕ is loxodromic on $c\mathcal{S}_n$ if and only if ϕ is iwip atoroidal.

Theorem (Dowdall-Taylor)**.** $\Gamma \leq Out(F_n)$ *a finitely generated subgroup.* $\Gamma \overset{qi}{\hookrightarrow} \mathcal{F}_n$ *and* Γ *is purely atoroidal if and only if* $\Gamma \overset{qi}{\hookrightarrow} c\mathcal{S}_n$*, where both maps are the orbit maps.*

Corollary. $\Gamma \overset{qi}{\hookrightarrow} c\mathcal{S}_n$ *implies* E_Γ *is hyperbolic.*

3. Width

The notion of width was introduced by Kent and Leininger for hyperbolic surface group etensions. Consider $f : S \to S$ and the mapping torus



For $\alpha \subseteq S$ a curve let α^* be a geodesic representative in $S \times R$. Define $width(\alpha) = diam(p_{\mathbb{R}}(\alpha^*))$. For arbitrary curves the width is unbounded.

However, Kent-Leininger prove

$$\sup_{\alpha \text{ s.c.c.}} width(\alpha) < \infty$$

And a more general statement for hyperbolic surface group extensions.

Now in F_n we have the sequence

$$1 \to F_n \to G \xrightarrow{\rho} \Gamma \to 1$$

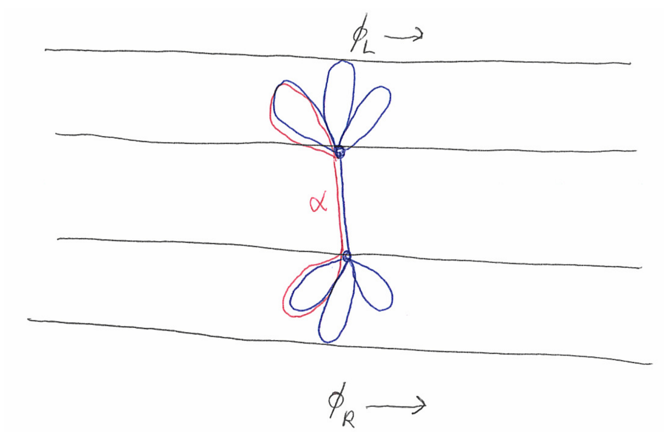
with G hyperbolic. If $a \in F_n$ let $a^* = [a^-, a^+] \subseteq G$ and define $width(a) = diam_\Gamma(\rho(\alpha^*))$.

Theorem. *When $\Gamma \leq Out(F_n)$, $\Gamma \overset{qi}{\hookrightarrow} c\mathcal{S}_n$ if and only if*

$$\sup_{a \text{ primitive}} width(a) < \infty$$

This theorem is an answer to our question "What do they look like?".

Lets return to the example $G_\Phi = F_6 \rtimes_\Phi \mathbb{Z}$, $\Phi = \phi_L \phi_R$.



Consider $\alpha_K = \phi_L^{-K} \phi_R^K(\alpha)$. Then $width(\alpha_K) \to \infty$.

4. A three manifold theorem

Theorem (Souto). *$f : S \to S$ closed $g \geq 2$*

$$rank(\pi_1(M_{f^n})) = 2g + 1$$

$n \gg 1$.

The key fact in the proof is due to Scott and Swarup: If $\Gamma < \pi_1(S)$ is finitely generated and infinite index then Γ is quasi-convex in $\pi_1(M_f)$.

- This fact generalizes to all hyperbolic extensions, by Dowdall-Kent-Leininger
- Generalized to bounded width hyperbolic extensions of F_n (Dowdall-Taylor, Mj-Rafi)

Theorem. *Suppose*

$$1 \to F_n \to E_\Gamma \to \Gamma \to 1$$

Hyperbolic bounded width. If $g_1, \ldots, g_k \in \Gamma$ *are infinite order with distinct endpoints. Set* $\Delta_m = \langle g_1^m, \ldots, g_k^m \rangle$*. For* $m \gg 1$*,*

$$rank(E_{\Delta_m}) = n + k$$

Corollary.

$$rank(F_n \rtimes_{\phi^m} \mathbb{Z}) = n + 1$$

for $m \gg 1$ *when* ϕ *is iwip and atoroidal.*