A MULTIPLICATIVE ERGODIC THEOREM FOR MAPPING CLASS GROUPS NOTES FROM THE OCTOBER 2016 MSRI WORKSHOP ON MAPPING CLASS GROUPS AND OUTER AUTOMORPHISM GROUPS

CAMILLE HORBEZ

This work is joint with Anders Karlsson.
Recall

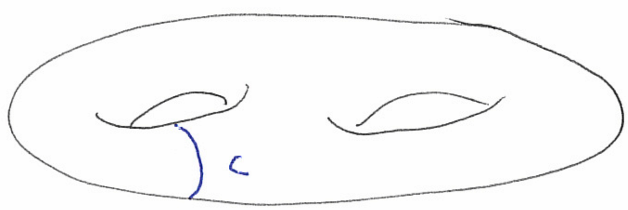
Theorem (Oseledets). *Let $(\Omega, \mathbb{P})$ be a standard probability space (e.g. $([0,1], \lambda)$). Suppose T acts on Ω ergodically and preserves the measure. Suppose $A : \Omega \to GL(N, \mathbb{R})$ is such that $\log^+ ||A(\cdot)|| \in L^1(\Omega)$. Consider*

$$\frac{1}{n} \log ||A(T^{n-1}\omega) \cdots A(T\omega)A(\omega)v||$$

There exist $\lambda_1 < \cdots < \lambda_k$ independent of ω such that almost surely there is a filtration $\{0\} = L_0(\omega) \subsetneq L_1(\omega) \subsetneq \cdots \subsetneq L_k(\omega) = R^n$ such that the limit of the above quantity is λ_i for $v \in L_i(\omega) \setminus L_{i-1}(\omega)$.

We want to investigate these processes in surfaces.

Given Σ a closed oriented hyperbolic surface with ρ the hyperbolic metric and c an essential embedded closed curve we have $||c||$ the shortest ρ-length of the curve c, and $||\phi(c)|| = \sup_c \frac{||\phi(c)||}{||c||}$.



We are interested in the growth of $||c||$ under an ergodic family of mapping classes.

Theorem (Horbez-Karlsson). *Suppose $(\Omega, \mathbb{P})$ is a standard probability space. Suppose T is an ergodic measure preserving action on Ω and $\Phi : \Omega \to Mod(\Sigma)$ satisfies $\log ||\omega(\cdot)|| \in L^1(\Omega)$. Then there exists $0 \leq \lambda_1 < \cdots < \lambda_k < \infty$ independent of $\omega \in \Omega$ such that almost surely there is a filtration*

Notes prepared by Edgar A. Bering IV.

$Y_1(\omega) \subsetneq Y_2(\omega) \subsetneq \cdots \subsetneq Y_k(\omega) = \Sigma$ *such that for all* c *isotopic into* Y_i *but not* Y_{i-1} *we have that*

$$\frac{1}{n} \log ||\Phi(T^{n-1}(\omega) \cdots \Phi(T\omega) \cdot \Phi(\omega)c|| \to \lambda_i$$

1. Examples

1.1. **Thurston.** Iteration of $\phi \in Mod(S)$. For all c we have $||\phi^n(c)||$ is either asymptotic to λ^n for finitely many λ, asymptotic to n, or bounded.

One proves this by first treating the case that ϕ is pseudo Anosov, so that all curves grow according to some single λ. In general pass to a rotationless power ϕ^m and apply the Nielsen-Thurston normal form.

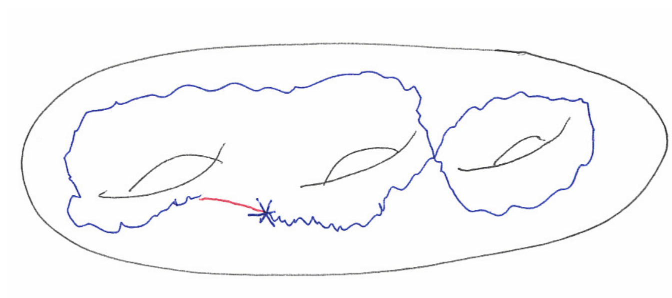
1.2. **Random Walks on** $Mod(\Sigma)$**.** Suppose μ is a probability measure on $Mod(S)$ and ϕ_i are independently identically distributed according to μ. Let $\Phi_m = \phi_m \cdots \phi_1$. (This is, in the framework of the theorem $\Omega = (Mod(\Sigma), \mu)^{\otimes \mathbb{N}}$ and $T =$ shift.)

Theorem (Karlsson)**.** *Assume* $\langle supp(\mu) \rangle = Mod(\Sigma)$ *and an* L^1 *condition. Then there exists a* $\lambda > 0$ *such that almost surely for all* n

$$\frac{1}{n} \log ||\Phi_m(c)|| \to \lambda$$

So the main theorem was already known for random walks.

1.3. **A natural ergodic process on** $Mod(S)$ **that is not a random walk.** Let M be a surface bundle over a surface



Pick a basepoint and run brownian motion on the base of M. Join the end point by a geodesic. The holonomy of this loop gives $\Phi_t \in Mod(\Sigma)$, an ergodic process in $Mod(\Sigma)$

Question (Open)**.** *Define growth rates of this* Φ_t *and relate them to the geometric invariants of* M

2. Remarks on the proof of the Main Theorem

Proposition (Burger-Pozzetti, Horbez-Karlsson-Bestvina)**.** *Every* $(\phi_n) \in Mod(\Sigma)^{\mathbb{N}}$ *has a subsequence* $\phi_{\sigma(n)}$ *such that there is a filtration* $Y_1 \subsetneq Y_2 \subsetneq \cdots \subsetneq Y_k = \Sigma$ *satisfying*

(1) *For all c, c' isotopic into Y_i but not Y_{i-1} there exists a K such that*

$$\frac{1}{K} \leq \frac{||\phi_{\sigma(n)}(c')||}{||\phi_{\sigma(n)}(c)||} \leq K$$

(2) *For all c isotopic into Y_i but not Y_{i-1} and c' isotopic into Y_{i-1}*

$$\frac{||\phi_{\sigma(n)}(c')||}{||\phi_{\sigma(n)}(c)} \to 0$$

This reduces the main theorem to showing that the above ergodic limits exist. This proposition will give the filtration and the finite list of exponents.

Proposition. *Under the hypotheses of the main theorem, almost surely for all c*

$$\frac{1}{n} \log ||\phi_n(\omega)(c)||$$

has a limit, with $\phi_n(\omega = \Phi(T^{n-1}\omega) \cdots \Phi(T\omega)\Phi(\omega)$.

Proof of the first proposition. Suppose $\rho \in Teich(\Sigma)$. $\phi_m^{-1}\rho$ converges projectively to a measured lamination ξ in Thurston's compactification. Therefore there exists a sequence λ_k such that for all c

$$\frac{1}{\lambda_k}||c||_{\phi_m^{-1}\rho} = \frac{1}{\lambda_k}||\phi_m(c)||_\rho \to i(c, \xi)$$

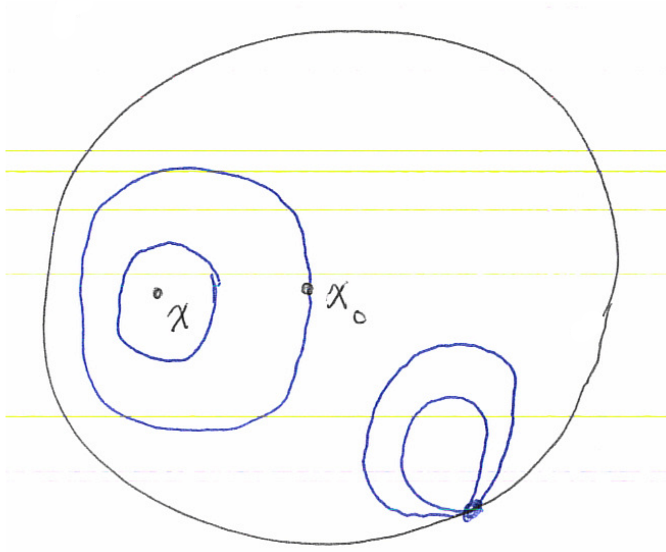
All curves on Σ that intersect ξ have the same growth type. Need to understand the growth of curves in $\Sigma \setminus supp\xi = A$.

The idea is that we can replace ϕ_n by $\phi_n \circ \psi_n$ with ψ_n supported on A so that $(\phi_n \circ \psi_n)^{-1}\rho \to \xi_2$ disjoint from A. How do we find ψ_n? Pick c_0 in Σ filling A. Choose ψ_n to minimize $||\phi_n \circ \psi_n(c_0)||$. The claim is that ξ_2 is not supported on A and we prove this with Sela's shortening argument. □

2.1. **Ideas in the proof of the second proposition.** .

The first is a theorem of Karlsson-Ledhappier. If G acts on X then ∂X the horoboundary of X has a good action. In our case we will use $G = Mod(\Sigma)$ and $X = (Teich(\Sigma), d_{Thurston})$.

Recall the Horoboundary of X. There is an inclusion $X \hookrightarrow C(X)$ by $x \mapsto (z \mapsto d(z, x) - d(x_0, x))$ for a fixed x_0. The closure of the image of X is $\bar{X}^{horo}$ the points are horospheres



Theorem (Karlsson-Ledhappier). *Suppose* $(\Omega, \mathbb{P})$ *is a standard probability space, and* T *an ergodic measure preserving map. Suppsoe* $g : \Omega \to G$ *and define* $g_n(\omega) = g(T^{n-1}\omega) \cdots g(T\omega)g(\omega)$. *Almost surely there is* $h_\omega \in \bar{X}^{horo}$ *such that*

$$\lim \frac{1}{n} d(x_0, g_n(\omega)^{-1} x_0) = \lim \frac{1}{n}(h_\omega(g_n^{-1}(\omega)x_0))$$

(The left hand quantity is called the drift.) In our setting, according to Walsh

$$\partial^{horo}_{Thurston} Teich(\Sigma) \cong PML$$

Theorem (Karlsson). *Almost surely there exists* $\xi_\omega \in PML$ *such that for all* c $i(c, \xi_\omega) > 0$

$$\frac{1}{n} \log ||\phi_n(\omega)(c)|| \to drift$$

The final plan of the proof is to use a larger compactification.

$$Teich(\Sigma) \hookrightarrow \prod_{A \subset \Sigma} \mathbb{PR}^{\mathcal{C}(A)}$$

Compactify here get $K = \overline{Teich(\Sigma)}$. There is a problem, K is not a horofunction bundary, but this can be worked around.