ℓ^2 TORSION OF FREE-BY-CYCLIC GROUPS NOTES FROM THE OCTOBER 2016 MSRI WORKSHOP ON MAPPING CLASS GROUPS AND OUTER AUTOMORPHISM GROUPS

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Think of ℓ^2 torsion as a notion of volume.

1. Motivation: Mapping Tori

Suppose $f : \Sigma \to \Sigma$ is a homeomorphism M_f the mapping torus.

Thurston tells us that $M_f = (\cup S_i) \bigcup (\cup H_i)$ where the S_i are Seifert fibered and the H_i are hyperbolic. This is unique, the JSJ decomposition.

Theorem (Kojima-McShane '14). *If $f : \Sigma \to \Sigma$ is pseudo Anosov then*

$$vol(M_f) \leq 3\pi|\chi(\Sigma)| \log \lambda(f)$$

Brooks showed that $||f||_{WP} \sim vol(M_f)$ and that $\log(\lambda(f)) \sim ||f||_{Teich}$. These are related already, but Kojima-McShane give an explicit constant.

Remark. There is no general lower bound.

Our goal is a similar statement about free-by-cyclic groups.

2. Background

Definition. *G is free by cyclic if*

$$1 \to F \to G \to \mathbb{Z} \to 1$$

In this case $G \cong F \rtimes_\Phi \mathbb{Z} = \langle F, t | t^{-1}xt = \Phi(x)\rangle$ with $\Phi \in Aut(F)$. This only depends on $\phi = [\Phi] \in Out(F)$. Denote by G_ϕ.

Definition (Lück). *ℓ^2-torsion is a χ-type invariant.*

The set up is $G = F$ or $\pi_1(S)$. $\Phi \in Aut(G)$. $\rho^{(2)}(G \rtimes_\Phi \mathbb{Z}) \in \mathbb{R}$ is the invariant.

Theorem (Lück-Schick '99). $M_f = \cup S_i \bigcup \cup H_i$

$$-\rho^{(2)}(\pi_1(M_f)) = \frac{1}{6\pi} \sum vol(H_i)$$

Our main theorem gives an upper bound on $-\rho^{(2)}(G_\phi)$. Lück showed that $-\rho^{(2)}(G_\phi) \geq 0$.

Question. *When is $-\rho^{(2)}(G_\phi) > 0$?*

Notes prepared by Edgar A. Bering IV.

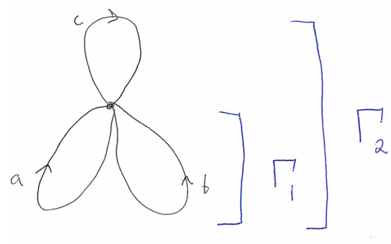
Recall. Relative Train Tracks (Bestvina-Handel). A map $f : \Gamma \to \Gamma$ where Γ is a graph with $\pi_1(\Gamma) = F$ is a RTT representative for ϕ if

(1) f is a homotopy equivalence inducing ϕ on $\pi_1(\Gamma)$
(2) $\{*\} = \Gamma_0 \subseteq \Gamma_1 \cdots \subseteq \Gamma_s = \Gamma$ such that $f(\Gamma_s) \subseteq \Gamma_s$.
(3) And three other properties

From an RTT get a transition matrix

$$[M(f)]_{ij} = \#\text{times } e_j^{\pm} \text{ appears in } f(e_i)$$

For example. $f : a \to ab, b \to ab^2, c \to cab^{-1}$



$$M(f) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

This matrix is in general lower block triangular. Let $M(f)_s$ be the block corresponding to $\Gamma_s \setminus \Gamma_{s-1}$. The relative traintrack property implies that $M(f^k)_s = M(f)_S^k$ where f^k is the tightened iterate.

If $M(f)_s$ is irreducible it has a Perron Frobenius eigenvalue $\lambda(f)_s$ Define the exponentially growing spectrum of f

$$\mathcal{EG}(f) = \{s | \lambda(f)_s > 1\}$$

3. Main Theorem

Theorem.

$$-\rho^{(2)}(G_\phi) \leq \sum_{s \in \mathcal{EG}(f)} n_s \log \lambda(f)_s$$

where $n_s = |\Gamma_s - \Gamma_{s-1}|$.

Corollary. *If* $f : \Gamma \to \Gamma$ *is irreducible then*

$$-\rho^{(2)}(G_\phi) \leq 3|\chi(F)| \log \lambda(\phi)$$

Some more about torsion. It is the determinant of an acyclic chain complex. Suppose C_* is a chain complex of finite dimensional vector spaces. Acyclic means we have a decomposition

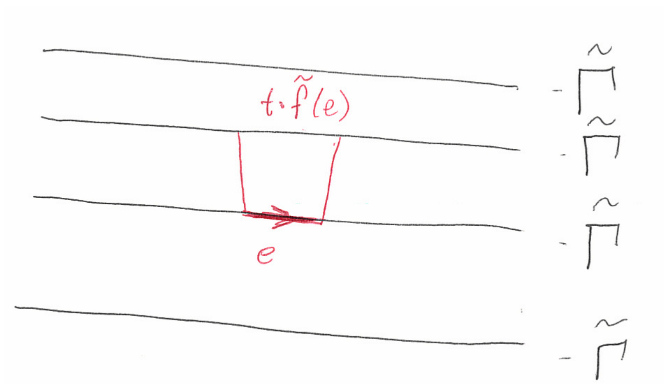
$$\begin{array}{ccccccccc}
 & & A_{n+1} & & A_n & & A_{n-1} & & \\
 & & & \searrow^{\sim} & & \searrow^{\sim} & & & \\
\cdots & & \oplus & & \oplus & & \oplus & & \cdots \\
 & & & & & & & & \\
 & & B_{n+1} & \xrightarrow{0} & B_n & \xrightarrow{0} & B_{n-1} & & \\
 & & \downarrow{\scriptstyle equals} & & \downarrow{\scriptstyle equals} & & & & \\
 & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow &
\end{array}$$

There is a formula

$$\log \rho(C_*) = \sum (-1)^{n-1} \log |\det(A_n \xrightarrow{\sim} B_{n-1}|$$

(calculated using an orthonormal basis when infinite rank).

Apply torsion to the chain complex of the universal cover of $X_f = \Gamma \times [0,1]/\sim$.



Let $A = \mathbb{C}[G_\phi]$, E the number of edges and V the number of vertices of Γ. From the diagram we see that the 2-cells are parameterized by edges, and so we have an acylic chain complex

$$0 \to A^E \to A^E \oplus A^V \to A^V \to 0$$

The boundary map is $[e_i - t\tilde{f}(e_i)] \oplus \partial e_i$. The left summand is the horizontal boundary, ∂_h. From Lück we get that

$$-\rho^{(2)}(G_\phi) = \log\det(\partial_h : \bar{A}^E \to \bar{A}^E)$$

where this determinant is the Fuglede-Kadison determinant in the functional analytic sense and $\bar{A} = \ell^{(2)}(G_\phi)$.

∂_h is right multiplication by $I - tJac(f) \in Mat_E(\mathbb{Z}[G_\phi])$

$$[Jac(f)]_{ij} = \text{coefficient of e in } \tilde{f}(e_i) \in \mathbb{Z}[F]$$

note that $[Jac(f)]_{\ell^1} = M(f)$, where we are replacing entries with their ℓ^1 norm. Therefore $Jac(f)$ has a block decomposition coming from the relative train track structure. Denote these blocks $Jac(f)_s$. This gives us the formula for torsion

$$-\rho^{(2)}(G_\phi) = \log\det(I - tJac(f)) = \sum_{s=1}^{S} \log\det(I - tJac(f)_s)$$

This equation is why we can work with ease with general automorphisms.