FAST NIELSEN-THURSTON CLASSIFICATION NOTES FROM THE OCTOBER 2016 MSRI WORKSHOP ON MAPPING CLASS GROUPS AND OUTER AUTOMORPHISM GROUPS

DAN MARGALIT

This work is joint with Strenner and Yurrtas.

1. Nielsen-Thurston Classification

Any $f \in Mod(S)$ is either

- periodic
- recuicble - fixes a multicurve
- pseudo Anosov

Theorem (Margalit-Strenner-Yurrtas)**.** *There is an algorithm that determines the Nielsen-Thurston type of a mapping class and finds period, reducing curves, stretch factor/invariant foliation in polynomial time with respect to work length in a generating set of twists. For just the Nielsen-Thurston type this algorithm is quadratic.*

A related recent theorem

Theorem (Bell-Webb)**.** *There is a polynomial time algorithm for Nielsen-Thurston type and translation distance in $\mathcal{C}(S)$.*

2. History

Dehn: An algorithm to find a finite collection of curves whose images determine a mapping class, and that a finite order class has order at most $4g-2$. So one can detect periodic automorphisms easily.

Bell, Koberda-Mangahas: An exponential upper bound for the reducible case.

Bestvina-Handel: General case. Kapovich with an appendix by Bell give an exponential upper bound.

Mosher, Agol: Given a pseudo Anosov find the foliation.

Calvez B_n Nielsen-Thurston in quadratic time.

Notes prepared by Edgar A. Bering IV.

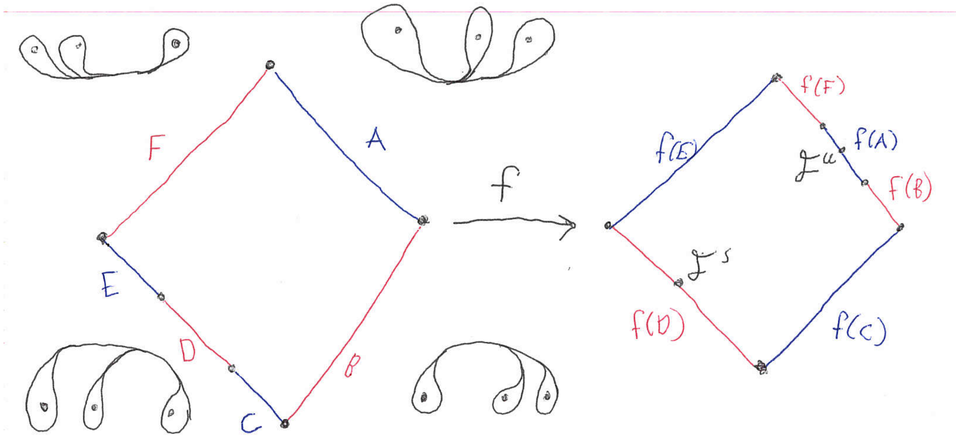
3. Basic Strategy

Iteration to find eigenvalues.

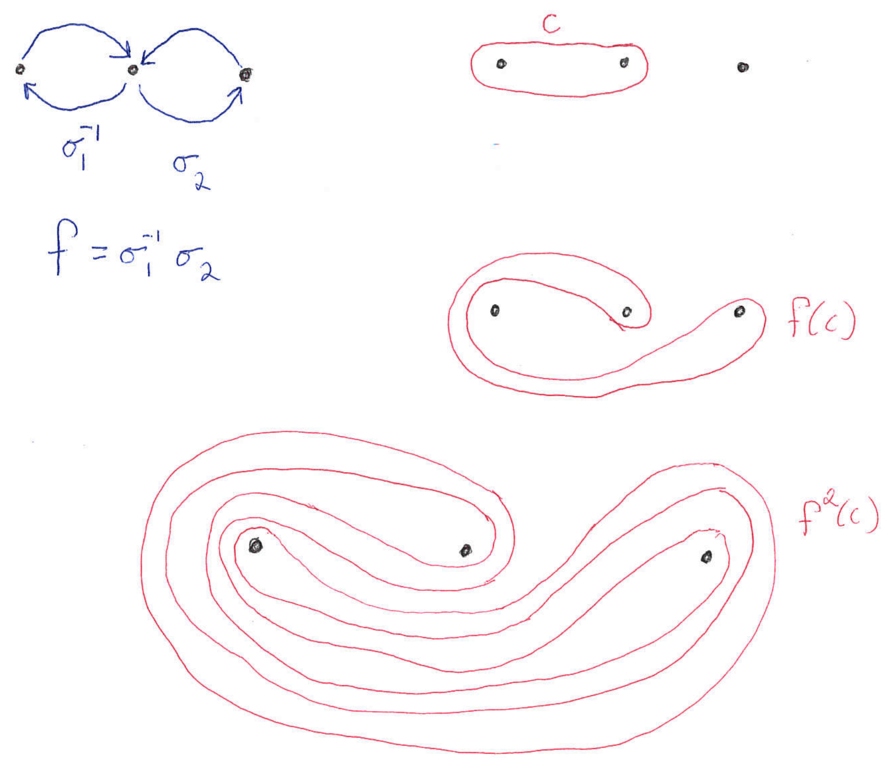
Thurston in his original announcement says iteration is "rather quick". Problem, Bell-Schleimer give f_k with spectral ratio f_k $1 + 14\phi^{-k}$ the golden ration. This is a family with exponentially slow eigenvalue convergence.

Brute-Force algorithm by Mosher, in a part of the handwritten predecessor to the Casson-Bleiler notes. Written in Hamidi-Tehrani-Chen '96. f acts on $\mathcal{MF}$ measured foliations as a piecewise integral linear homeomorphism. Search for fixed points in $\mathcal{PMF}$ or just eigenvalues. In the worst case f has exponentially many linear pieces, so this approach is exponential.

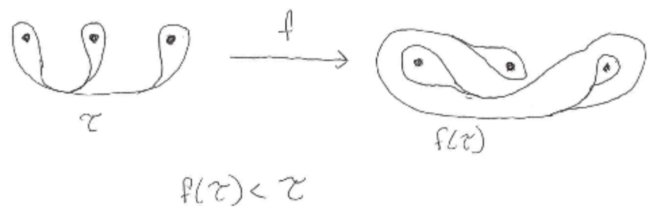
4. Famous Example



This is a picture of $\mathcal{PMF}(S_{0,4})$ in train track charts, illustrating also the PL decomposition of the "famous example" illustrated below



The problem can be difficult. In this example we are lucky because the fixed points of f are far away, the "hard part" is when they are close.



If we have an invariant train track τ then $\mathcal{PMF}(\tau)$ is contained in the linear basin of the unstable foliation.

The main idea is that we only need a matrix representative of f at one point of the linear basin. So we will find the linear basin quickly by iteration.

5. Toby Hall's Method

This is software for braids. It is very fast compared to Xtrain that runs the Bestvina-Handel method. The scheme

- Iteratively compute the action of f on $f^k(c)$
- This gives a sequence of matrices
- Since a pseudo Anosov has source-sink dynamics (M_k) is eventually constant (for experts, periodic).

- The stable matrix M_N is in the linear basin $\lambda = \sigma(M_N)$ and $\mathcal{F}^u$ is the λ eigenvector.

Question (From the audience). *How do you compute M_k?*

Fix generators for $MCG(S)$ and compute their entire PL linear representatives. Use these representatives to calculate $f^k(c)$, points and M_k from matrix multiplication. This method is also present in Bell's Flipper.

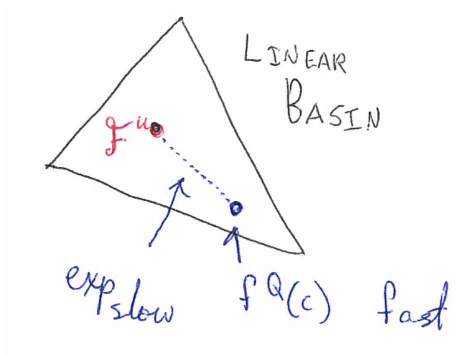
6. Main Theorem

Theorem (Margalit-Strenner-Yurrtas). *Fix S. Fix a cell decomposition $\tau = \{\tau_1, \ldots, \tau_n\}$ of $\mathcal{PMF}(S)$. Fix c a vertex cycle of some τ_i. In the pseudo Anosov case there exists Q depending on S, τ such that for all pseudo Anosov $f \in Mod(S)$, $f^Q(c)$ is in the linear basin of f.*

This implies the existance of the algorithm.

Question (Audience). *What about the Bell-Schleimer examples?*

Take Bell-Schleimer map f_k for $k = 10,000$, this is the picture



The algorithm in the pseudo Anosov case is

- Compute $f^Q(c)$
- Compute M for f acting on $\mathcal{PMF}(S)$ at $f^Q(c)$.
- Compute the largest eigenvalue and corresponding eigenvector of M

The generalized algorithm uses a larger $Q' > Q$, first check if M has an eigenvalue 1, if so it is reducible. Otherwise proceede to compute the largest eigenvalue.

Remark that $Q \sim g^2 + p + b$ with precise constants pending.

7. The proof in six parts (pA case)

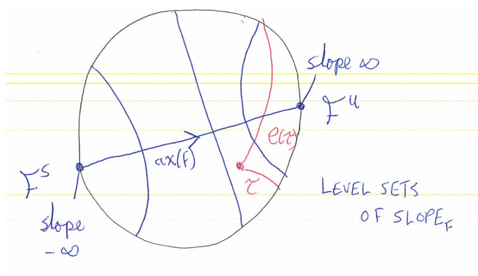
7.1. **Gadre-Tsai constant.** There is a number $K = K(s)$ that implies f^K has positive translation length in $\mathcal{C}(S)$.

7.2. **Slopes.**

Definition.

$$Slope_f(x) = \log \frac{i(x, \mathcal{F}^s)}{i(x, \mathcal{F}^u)}$$

Can extend this from curves to x is either a measured foliation or train track by taking the max over vertex cycles.



Remark that the choice of measure is unimportant, changes of measure are translations of the slope.

7.3. **Fact about slopes.**

$$Slope_f(f^n(c)) = Slope_f(c) + 2n \log \lambda$$

7.4. **Slope lemma.**

Lemma. *$Slope_f(c) \geq Slope_f(\tau) + 2K \log \lambda$ and $\mathcal{F}^u \leq \tau$ implies $c < \tau$*

7.5. **Proposition on intersection.**

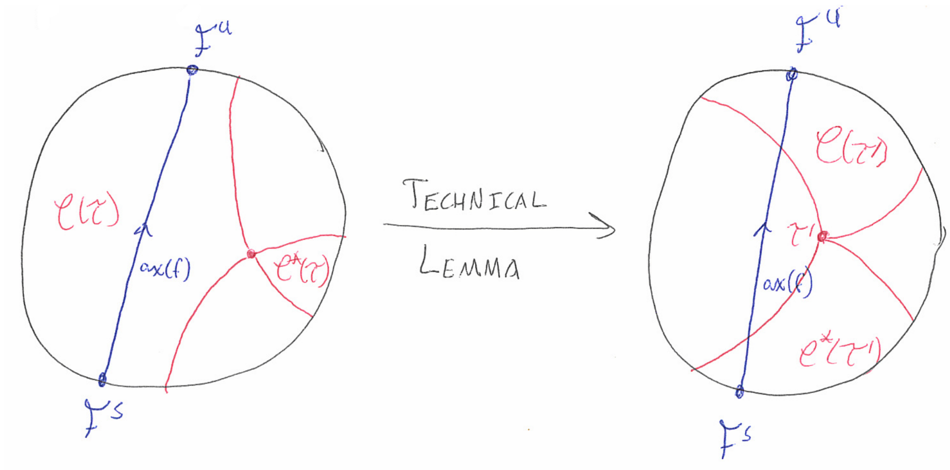
Lemma. *$\mathcal{F}^u < \tau$ and $\mathcal{F}^s \pitchfork \tau$ implies that $f(\tau) \leq \tau$ (this might require a power)*

7.6. **Technical Lemma.**

Lemma. *If $\mathcal{F}^u < \tau$ then one can split τ to τ' so that $\mathcal{F}^u < \tau', \mathcal{F}^s \pitchfork \tau'$ and $|Slope_f(\tau) - Slope_f(\tau')| \leq 8K \log \lambda$.*

8. Proof of the main theorem from the pieces (pA case)

Given f, let $\tau = \tau_i$ some chart such that $\mathcal{F}^u < \tau$.



The Fact+Lemma imply that $f^{6K}(c) < \tau'$ the proposition implies $f^{6K}(c)$ is in the linear basin.

9. Questions

- Can we compute the minimal stretch factors in small genus now that we have this fast algorithm?
- Can we investigate what algebraic degrees are possible?
- Can we experimentally understand the genericity of pseudo Anosovs?
- Is there a lower bound on Q?