GEOMETRY OF HYPERBOLIC GROUPS AND FIX POINT PROPERTIES NOTES FROM THE OCTOBER 2016 MSRI WORKSHOP ON MAPPING CLASS GROUPS AND OUTER AUTOMORPHISM GROUPS

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Throughout the talk we will look at G finitely generated from the view-point of isometric (non-linear) actions on Banach spaces.

Masur-Ulam: Every isometry of a Banach space is affine $v \mapsto Uv + b$ with U unitary.

Recall Property (T), it is equivalent to every action by isometries on a Hilbert space has a global fixed point (FH).

Recall a-T-amenability, there exists a proper action on a Hilbert space such that $\forall v \in X, |g|_S \to +\infty \implies ||gv|| \to \infty$.

At present nothing is known for $Mod(\Sigma)$ a hyperbolic surface, $Aut(F_n), Out(F_n)$ with $n \geq 4$.

1. Test cases to guide us

1.1. Lattices of isometries of symetric spaces of non-compact type.

Symmetric space:	$\mathbb{H}^n_{\mathbb{R}}$	$\mathbb{H}^n_{\mathbb{C}}$	$\mathbb{H}^n_{\mathbb{H}}$	$\mathbb{H}^n_{\mathbb{O}}$	rk ≥ 2
Lattices:	a-T-amenable			(T)	

1.2. Hyperbolic Groups.

Random groups for density $d \in (1/3, 1/2)$ are hyperbolic and have (T). For density $d < 1/6$ are hyperbolic and a-T-amenable.

2. Stronger (T)/weaker a-T-amenable

2.1. Strengthening 1.

Consider affine actions on Hilbert space, not isometric but uniformly bilipschitz. That is $g \in G$, for all $v \in H$ $v \mapsto \pi(g)v + b_g$ and $sup||\pi(g)|| < +\infty$.

Higher rank lattices have the fixed point property for these actions. (Bader-Furman-Gelander-Monod, Shalom).

Conjecture (Shalom)**.** *Hyperbolic groups have proper actions of this type.*

Notes prepared by Edgar A. Bering IV.

2.2. **Strengthening 2.** Replace Hilbert space by $L^p(X,\mu)$ with $p \in [1,\infty)$. Note that for all p $(FL^p) \Rightarrow (T)$.

a-T-amenable implies a-FL^p-amenable. The converse is also true for $p \in [1,2]$ (Bader-Gelander-Monod).

For $p \gg 2$ FL^p is strictly stronger than (T) and a-FL^p-amenability is strictly weaker than a-(T)-amenability.

According to Boudon, Boudon-Pajot, Yu, Nica. Every hyperbolic group acts properly on some L^p if $p >$ conformal dimension $\partial_\infty G$. In fact G hyperbolic with (T) are a-FL^p-amenable when $p > cd(\partial_\infty G$.

Methods

- Yu uses Mineyev's construction of "unitary tangent bundle" for hyperbolic groups.
- Nica uses conformal structure of $\partial_\infty G$ (Mineyev)
- Boudon $p > cd(\partial_i nftyG)$ implies there exists d a metric on $\partial_\infty G$ that is quasiconformally equivalent to a visual metric and the Hausdorff dimension of $(\partial_\infty G, d) < p$. Boudon constructs a proper cocycle such that for all u on $\partial_\infty G$ Lipschitz there exists f_u on G with $f_u \in \ell_p(G)$.

Remark that FL^p implies stronger rigidity results (Navas).

A way to measure "how strong a (T)-property a group has" is to consider

$$\mathcal{F}(G) = \{p \in [1,\infty) : G \text{ has } FL^p\}$$

when G has (T).

We know $[1,2] \subseteq \mathcal{F}(G)$.

- $\mathcal{F}(G)$ is open (Fisher-Margulis, Drutu-M. Kapovich)
- If G is a higher-rank lattice, $\mathcal{F}(G) = [1,\infty)$
- If G is hyperbolic, $\mathcal{F}(G) \subseteq [1, cd(\partial_\infty G)$

Question (Boudon)**.** *Do there exist $\mathcal{F}(G)$ bounded containing $[1,p_0]$ when p_0 is large?*

Question (Chatterji-Drutu-Haglund)**.** *Does there exist a connection between $p(G)$ and $cd(\partial_\infty g)$?*

The first question is answered in the positive by Noor-Peres, who give a sequence of hyperbolic groups answering it.

To approach question 2 the correct setting is random groups.

3. Triangular Model of Random Groups

Consider $d \in (0,1)$ the density parameter, $m \in \mathbb{N}$ for all $m \in \mathbb{N}$ we can define S_m an abstract generating set $|S_m| = m$. Let W_m be the set of words in $S_m^{\pm}$ cyclically reduced of length 3. $|W_m| = (2m)^3$ Pick $R \subseteq W_m$, $|R| = (2m)^{3d}$ uniformly independently at random. The model $\mathcal{M}(m,d)$ is $G = \langle S_m | R \rangle$.

A property P occurs asymptotically almost surely in this model if

$$\mathbb{P}(\Gamma = \langle S_m | R\rangle \text{ has } P) \to 1$$

For example

- $d > 1/2$ implies $G \cong 1$ or $\mathbb{Z}/2\mathbb{Z}$ aas
- $d < 1/2$ implies G is hyperbolic aas (Zuck)
- $d > 1/3$ implies G has (T) aas (Zuck). (Kotowok-K)
- $d < 1/3$ implies G is free aas (Autoniuk-Luczak, Swiatkowski)

Theorem (Drutu-Mackay)**.** *For all $d > 1/3$ there exists a C such that aas G has FL^p for*

$$p \in \left[1, c\left(\frac{\log m}{\log\log m}\right)^{1/2}\right]$$

Corollary. *For all $d \in (1/3, 1/2)$ there exists C, k depending on d such that aas*

$$c\left(\frac{\log m}{\log\log m}\right)^{1/2} \leq p(G) \leq cd(\partial_\infty G) \leq K\log m$$

This implies

$$cd(\partial G)^{1/2-\epsilon} \leq p(G) \leq cd(\partial G)$$

Remark all results are true for actions uniformly Lipschitz with constant $L \leq 2^{\frac{1}{2p}}$

4. Gromov Density Model

Fix a density $d \in (0,1)$ and an alphabet S with $|S| = k$. Consider W_ℓ all reduced cyclically reduced words of length ℓ in $S^{\pm}$. $|W_\ell| \asymp (2k)^\ell$. Pick $R \subseteq W_\ell$, $|R| = (2k)^{\ell d}$ uniformly independently at random. The model is $\mathcal{G}(k,d) = \langle S|R\rangle$. We have a similar definition of aas as ℓ goes to infinity.

We know

- $d > 1/2$ then $G = \{1\}$ or $\mathbb{Z}/2\mathbb{Z}$ aas
- $d < 1/2$ then G is hyperbolic aas (Gromov, Ollivier)
- $d > 1/3$ then G has (T) aas (Zuck, Gromov, Ollivier)
- $d < 1/6$ then G is a-T-amenable aas (Ollivier, Wise)
- $d < 5/24$ then G does not have (T) (Mackay, Przytycki)

Theorem (Drutu-Mackay)**.** *For all $p_0 \in [1, +\infty)$, for all $K > 10 \cdot 2^{p_0}, d > 1/3$ aas in the Gromov model G has FL^p with $p \in [1, p_0]$.*