TAME HIERARCHIES FOR CURVE GRAPHS NOTES FROM THE OCTOBER 2016 MSRI WORKSHOP ON MAPPING CLASS GROUPS AND OUTER AUTOMORPHISM GROUPS

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Let's give a shout out to the super-star of this workshop: the Curve Graph. This talk will also concern it. Masur and Minsky showed that the curve graph is hyperbolic.

Definition. *Let X be a metric space. X has* asymptotic dimension n *($asdim(X) = n$) if for all R there exists a covering of X by uniformly set such that every ball of radius R intersects $n+1$ sets.*

Examples

(1) $asdim(\mathbb{R}) = 1$
(2) Trees $asdim(T) = 1$
(3) If $X \to Y$ is a quasi-isometric embedding then $asdim(X) \leq asdim(Y)$.
(4) This implies that if X is a quasi-tree $asdim(X) = 1$
(5) And that groups have well-defined asymptotic dimension
(6) $asdim(X \times Y) \leq asdim(X) + asdim(Y)$

Theorem (Bestvina-Bromberg-Fujiwara)**.** *$asdim(Mod(S)) < \infty$ for S closed of genus $g \geq 2$*

This is a consequence of

Theorem (Bestvina-Bromberg-Fujiwara)**.** *$Mod(S)$ admits an equivariant quasi-isometric embedding into a product of quasi-trees.*

Some history. Bell-Fujiwara '08 showed that $asdim(\mathcal{C}(S)) < \infty$. Bestvina and Bromberg improved this to $asdim(\mathcal{C}(S)) \leq 4g - 4$.

Remark The theorem above is not what Bestvina, Bromberg and Fujiwara prove. They prove $Mod(S) \stackrel{qie}{\to} \prod$ hyperbolic spaces where each hyperbolic space is a quasi-tree of spaces with uniformly bounded asymptotic dimension.

The product of quasi-trees improvement is due to Hammenstädt.

Definition (Hyperbolic relatively hyperbolic graphs)**.** *G a hyperbolic graph and $\mathcal{H} = \{H_c | c \in C\}$ a family of subgraphs of G. We say G is relatively hyperbolic relative to $\mathcal{H}$ if*

Notes prepared by Edgar A. Bering IV.

- *H_c is connected and uniformly quasi-convex. That is $H_c \to G$ is an L-quasi-isometric embedding and L is independent of c.*
- *For $c \neq u$ the diameter of the shortest distance projection $H_c \to H_u$ is uniformly bounded.*
- *$\mathcal{H}$-electrification: cone off each H_c to get a graph $\mathcal{EG}$ which is hyperbolic.*

Theorem (Bestvina-Bromberg-Fujiwara)**.** *G quasi-isometrically embeds into the product $\mathcal{EG} \times V$ where V is a quasi-tree of spaces H_c. Moreover if $asdim(H_c) \leq n$ uniformly then $asdim(G) \leq asdim(\mathcal{EG}) + n + 1$, and if the H_c's are quasi-trees then V is a quasi-tree.*

Example. Take Σ of genus $g \geq 2$ with $m \geq 0$ punctures. The graph of non-separating curves in S, $\mathcal{C}^\circ(S) \subseteq \mathcal{C}(S)$. When $m \leq 1$ the inclusion is a quasi-isometric embedding. When $m = 2$ this is no longer true, we can find curves of arbitrary distance in $\mathcal{C}^\circ(S)$ that are distance 2 in $\mathcal{C}(S)$.

We can still understand the geometry of $\mathcal{C}(S)$. Let C be the family of separating curves which decompose S into S_c of type $(g, 1)$ and a pair of pants. Take H_c to be the non-separating curves in S_c. The electrification $\mathcal{C}^\circ(S)$ with respect to these H_c is $\mathcal{C}(S)$, that is $\mathcal{C}^\circ(S)$ is hyperbolic rel $\{H_c\}$ so has finite asymptotic dimension.

Definition (Hierarchy of hyperbolic graphs)**.** *A finite collection of graphs $G_1, \ldots, G_k$ that are hyperbolic satisfying*

1. *$G_k = G$*
2. *G_{i+1} is hyperbolic relative to a family $\mathcal{H}_i$ of subgraphs and electrifies to G_i.*
3. *G_1 is hyperbolic.*

We say such a hierarchy is *tame* if $asdim(G_1)$ and all graphs in $\mathcal{H}_i$ is finite. Implies $asdim(G_k) < \infty$.

If G_1 is a quasi-tree and all $\mathcal{H}_i$ are quasi-trees then G_k embeds into a product of k quasi-trees.

Theorem. *$\mathcal{C}(S)$ admits a hierarchy of depth $4g - 4$ by quasitrees. Hence $asdim(\mathcal{C}(S)) \leq 4g - 4$.*

Question. *Does the free factor graph admit a hierarchy of quasitrees?*

Fact. The free splitting graph admits a hierarchy of relative free-factor graphs. We digress a moment about the difficulty of this conjecture before returning to the theorem. Recall that the free splitting graph is the geometric complex with vertices isotopy classes of embedded spheres in $M = \#S^1 \times S^2$ with simplices for disjointness. When cutting to build the hierarchy we run into sphere complexes of connect sum manifolds with marked points, which have infinite asymptotic dimension. So the naïve approach won't work here.

Back to the theorem. Work with the geometric complex G with simple closed curves as vertices, connecting c, d if and only if there is a component of

$S \setminus c \cup d$ which is neither a quadrangle nor a hexagon. This is a locally infinite $Mod(S)$ graph. Claim G is hyperbolic, this follows from Kapovich-Rafi and Masur-Minsky.

Remains to show that G is an infinite diameter quasi-tree and a good base for our construction, but we are out of time.