<u>MSRI - CAT(0)</u>

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On dense subgroups of simple Lie groups.
w/ N. de Saxcé

§1 Dense subgroups.

notation:

G : simple Lie group

$m = \dim(G)$

eg $G = SU(2)$, $SL(2,\mathbb{R})$

- We care about growth of <u>(Hausdorff) dimension</u>

$$\dim_H(T) = \inf\{\alpha > 0 : \mathcal{H}^\alpha(T) < \infty\}$$ uniform bound

where $$\mathcal{H}^1(T) = \liminf_{n \to 0} \left\{ \sum_{i \geq 1} \operatorname{diam}(B_i) \;\middle|\; \begin{array}{l} \operatorname{diam}(B_i) \leq n \\ T \text{ is covered by } B_i \\ (\text{ie } T \subset \bigcup_i B_i) \end{array} \right\}$$

and in general

$$\mathcal{H}^\alpha(T) = \liminf_{n \to 0} \left\{ \sum_{i \geq 1} \operatorname{diam}(B_i)^\alpha \;\middle|\; \begin{array}{l} T \subset \bigcup B_i \\ \text{and} \\ \operatorname{diam}(B_i) \leq n \end{array} \right\}$$

<u>Examples</u>

T	$\dim_H(T)$
interval	1
square	2
Cantor set	$\frac{\log 2}{\log 3}$.

<u>Main Thm</u>: If $T \lneq G$ is Borel measurable, and <u>dense</u> then $\dim_H(T) \neq m = \dim_H(G)$.

Previous result:

<u>de Saxcé</u>: $\dim_H(T) \notin (0, m)$ for T as in main theorem.

Hence, $\dim_H(T) = 0$.

($\because$ $\dim_H(T) \notin (0, m]$)

Applications of Main Thm:

Q1: Does $\exists$? Γ NOT-Borel measurable with $\dim_H(\Gamma) = m$?

A: Yes

construction: take $\Gamma = SL(2,K)$ — subfield of $\mathbb{R}$ not-Lebesgue measurable

Q2: Does $\exists$? Γ: uncountable with $\dim_H(\Gamma) = 0$?

A: Yes

take $\Gamma = SL(2,K)$ — uncountable subfield of $\mathbb{R}$ s.t. $\dim_H(K) = 0$

Q3: Does $\exists$? $\Gamma \subset$ Torus $= \mathbb{R}/\mathbb{Z}$ a Borel measurable subgroup (nec. abelian) s.t. $\dim_H(\Gamma) = \alpha$ for some $\alpha \in (0,1)$?

* Rk at end for why cannot reuse above idea *

A: Yes

take $T_n = \{ x \in \mathbb{R}/\mathbb{Z} : d(x, (2^{-n^n}\mathbb{Z}/\mathbb{Z})) \leq n \cdot 3^{-n^n} \}$ $\forall n \geq 1$

and $\Gamma = \bigcup_{n \geq} T_n$

- $\dim_H(T_n) = \dfrac{\log 2}{\log 3}$

$\Rightarrow \dim_H(\Gamma) = \dfrac{\log(2)}{\log(3)} \in (0,1)$.

§2 Continuous functions on G

φ, ψ will denote conti functions on G with compact support ($\in C_c^0(G)$)

convolution

$$(\varphi * \psi)\, g := \int_G \varphi(gx^{-1})\,\psi(x^{-1})\, dx.$$

Main Thm 2: $\exists\, p \geq 1$ s.t. $\forall \varphi \in C_c^0(G)$ and $\varphi^{*p} \in C_c^1(G)$ — higher regularity

Q4: $\exists$? $\varphi \in C^0(\text{Torus})$ s.t. $\forall p \geq 1$ $\varphi^{*p} \notin C^1(\text{Torus})$

A: Yes.

Pick φ with Fourier coefficients $c_n(\varphi) \notin O(n^{-\varepsilon})$

for instance, $\varphi(x) = \sum_{k \geq 1} \frac{1}{k^2} e^{i\pi 2^k x}$.

$c_n((\varphi^{*p})') = 2i\pi n\, c_n(\varphi)^p$. not bounded.

§3 Convolution of measure

a measure $\mu \in \mathcal{M}_c(G)$ is called

<u>α-Frostman</u> provided $\exists\, c > 0$ s.t.

$$\mu(B(x, r)) \leq c \cdot r^{\alpha} \qquad \forall x \in G,\ \forall r \leq 1$$

<u>Rk1</u>: any such μ is 0-Frostman

<u>Rk2</u>: if μ has bounded density then μ is m-Frostman

<u>Main Thm 3</u>: $\exists\, \alpha < m$ and $p \geq 1$ s.t. for any α-Frostman measure μ the convolution μ^{*p} has C^1-density

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that is $\mu^{*p} = \varphi \, \underbrace{dg}_{\text{Haar measure}}$ and $\varphi \in C^1(G)$

<u>Frostman</u> (Lemma)

$$\dim_H(T) = \sup \left\{ \alpha \geq 0 \ : \ \exists\ \alpha\text{-Frostman measure } \mu \text{ s.t. } \mu(T) \neq 0 \right\}$$

<u>Rk</u>: Uses T: Borel measurable here

<u>Rk</u>: Thm 3 $\Rightarrow$ Thm 1

since $\mu = \varphi\, dg$ is n-Frostman

$\Rightarrow$ α-Frostman for all smaller $\alpha < n$.

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§4 proof of Thm. 3

in special case: $G = SU(2)$ (proof generalizes to G: <u>compact</u>)

$(\pi_n, V_n) \coloneqq$ irreducible representation of G of dimension $= n$.

s.t. $\pi_n\left(\begin{bmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{bmatrix}\right) = \operatorname{diag}\left(e^{i(n-1)\theta}, e^{i(n-3)\theta}, \ldots, e^{-i(n-1)\theta}\right)$

<u>Plancherel's Thm</u>:

$$L^2(G) \xrightarrow{\ \simeq\ } \widehat{\bigoplus_{n \geq 1}} (V_n)^n$$

is an isomorphism as unitary representations of G.

in particular, $\forall \varphi \in L^2(G)$

$$\|\varphi\|^2_{L^2} \geq n \, \|\pi_n(\varphi)\|^2 \qquad \text{(more familiar formulation)}$$

where $\pi_n(\varphi) = \int_G \varphi(g)\, \pi_n(g)\, dg$

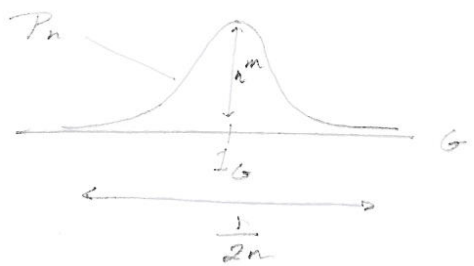
$\in \operatorname{End}(V_n)$

here $\mu = \varphi\, dg$.

Proposition: $\|\pi_n(\mu)\| = O(n^{-3})$

take $P_n \in C_c^0(G)$ s.t. $\int_G P_n(g)\,dg = 1$.

$1_G = e$



computation:

$$\|\pi_n(P_n * \mu)\| \le \|P_n * \mu\|_{L^2}$$
$$\ll \|P_n * \mu\|_{L^\infty}$$
$$\ll \|P_n\|_{L^\infty} \cdot \sup_{x \in G} \mu(B(x, \tfrac{1}{n}))$$
$$\ll n^m \cdot n^{-\alpha}$$

Hence, $\|\pi_n(P_n * \mu)\| = O(n^{-\varepsilon})$

$$\|\pi_n((\delta_e - P_n) * \mu)\| \le \|1 - \pi_n(P_n)\| \, \|\pi_n(\mu)\|$$

$\mathrm{supp}(P_n) = \frac{1}{2n} \Rightarrow |\theta| \le \frac{1}{4n} \cdot |e^{ik\theta}|$

$$\le \frac{1}{2}$$
$$\le \frac{1}{2}\|\pi_n(\mu)\|$$

$\forall |k| \le n \Rightarrow \|\pi_n(\mu)\| \in O(n^{-\varepsilon})$

Rk: when G: NOT compact.

need Fourier coefficients to decay sufficiently quickly:

Proposition

$\mathcal{U}$ nbhd of $e \in G$.

$\forall F \in L^2(\mathcal{U})$

$$\|\mu * F\| \ll \underbrace{\|P_n * F\|_{L^2}}_{} + n^{-\varepsilon}\|F\|_{L^2}$$

contribution from low-frequency (not killed by P_n)

B.I.G (initials): $\forall F \in L^2(\mathcal{U})$

$$\int_G |\langle T_g F, F\rangle| \ll \|P_n * F\|^2_{L^2} + n^{-\varepsilon}\|F\|^2_{L^2}$$

Rk: μ^{*p} only makes sense when T is a group (need semigroup really)

DNE $A \subset \mathbb{R}$ Borel measurable s.t. $\dim_H(A) \in (0,1)$

by Edgar+Miller 2003