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Homotopy and homology complexity
in negative curvature
w/ Uri Bader
+
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negative curvature
topology $\longleftrightarrow$ Area
· Gauss Bonnet

Ballmann + Gromov + Schroeder : $\forall$ even dimensions d. $\exists$ constant $C_d > 0$ s.t. $\forall$ Riemannian d-manifolds, M, of sectional curvature in $[-1, 0)$.

$$\boxed{b_k(M) < C_d \cdot vol(M)}$$

b_k = rank $(H_k(M; \mathbb{Z}))$.

Main Thm 1 : $\forall\ d \neq 3$ $\exists$ constant $C_d > 0$

Same initials as above :)

s.t. $\forall$ Riemannian d-manifolds, M, with sectional curvature in $(-1, 0)$.

$$\boxed{\log |\mathrm{Tors}(H_k(M; \mathbb{Z}))| < C_d \cdot vol(M)}$$

Rk : this lets us count. Riem. mflds. of low volume.

$$\mathcal{P}_d^{htp}(V) = \#\{M : vol(M) < V\}/\text{homotopy}$$

$$\mathcal{P}_d^{homeo}(V) = \#\{M : vol(M) < V\}/\text{homeo.}$$

Main Thm 2

$d \geq 4 \Rightarrow \alpha \cdot V \log(V) < \log \mathcal{P}^{htp}(V) < \beta \cdot V \log(V)$

Further,

$d \geq 5 \Rightarrow$ same ineq. for $\mathcal{P}^{homeo}(V)$

Consequence : homotopy equiv $\Rightarrow$ homeo. for $d \geq 5$.

this generalizes work of Burger + Gelander + Mozes + Lubotzky

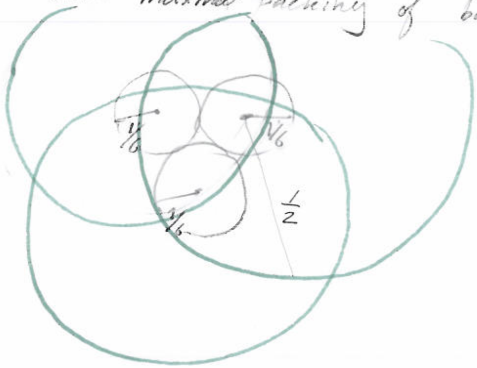
Idea : Find simplicial representatives ("nice") for homotopy types.

$\rightsquigarrow$ Nerves construction.

construction

Assume $\text{injrad}(M) \geq 1$.

M: take maximal packing of balls of radius $\frac{1}{6}$



look at balls of radius $\frac{1}{2}$ at these centers.

since $\text{injrad}(M) \geq 1$ all these balls are convex + contractible in M.

take $\mathcal{U}$ = cover by these open balls of radius $\frac{1}{2}$

the nerve encodes adjacency of these balls.

$M \simeq \text{nerve}(\mathcal{U})$ (when $\mathcal{U}$ is good)

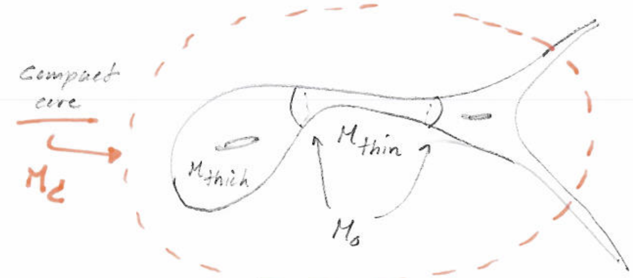
$$\#\text{vertices} = \underset{\text{number of balls}}{\#\mathcal{U}} < C \cdot \text{vol}(M)$$

$\text{degree}(v) < D$ (uniformly bounded)

* We say that $\text{nerve}(\mathcal{U})$ is a $(C\cdot\text{vol}(M), D)$ - simplicial complex.

Thick - thin decomposition.

M



$M_{thick} = \{ x : \text{injrad}(x) > \varepsilon_d \}$

ε_d : Margulis constant

Fact: $M_c \cong M$.

$M_{thick} = D^{d-1}$ - bundle over S^1.

$$\begin{tikzcd}
M_0 \arrow[r] \arrow[d] & M_{thin} \arrow[d] \\
M_{thick} \arrow[r] & M_c
\end{tikzcd}$$

$(d-2)$ connected.

$k < d-2 \overset{\text{Excision}}{\Longrightarrow} H_k(M) \cong H_k(M_c) \cong H_k(M_{thick})$

since $M_c = M_{thick} \cup_{M_0} M_{thin}$.

<u>Main Thm 2</u> : $\boxed{(M_{thick}, M_0) \overset{\text{hom. equiv.}}{\simeq} (C_d^{\text{vol}(M)}, D)}$ — simplicial pair.

"simplicial thick-thin decomposition"

<u>Rk</u> when $\exists$ bounds on $H_k(M_{thick})$ then recover (both) <u>BGS</u>. bounds, from Excision argument.

rel. homology.

$k \geq d-2 \geq 2$

Excision $\Rightarrow$ $H_k(M) \cong H_k(M_c)$

$\cong H_k(M_c, M_{thin})$

$\cong H_k(M_{thick}, M_0)$

$H(\partial M_c) = \langle \mu, \lambda \rangle$
torus meridian longitude

<u>Q</u>: Why do we need to exclude <u>dim = 3</u> ?

<u>Example</u>: M = hyperbolic knot complement.

$\downarrow$ Dehn fill. (glue in solid torus meridian $\longrightarrow \alpha \in \mathbb{Z}^2$)

M_α

Explicitly, $M = \underset{\text{compact core}}{M^c} \cup_{\partial M^c} \underbrace{T^2 \times [0,\infty)}_{\text{cusps}}$

For pushout diagrams:

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \downarrow \\ C & \hookrightarrow & D \end{array}$$

Excision $\Longrightarrow$ rel. hom. $H_k(A, C) \cong H_k(D, B)$

Torsion in these hyperbolic Dehn fillings can explode. $\rightsquigarrow$ obstacle to having Thm.

From Meyer-Vietoris:

$$H_1(M_\alpha) \cong \text{coker}\left(\underset{S^1 \times S^1}{H_1(\text{Torus})} \longrightarrow H_1\left(\begin{smallmatrix}\text{Solid}\\ \text{Torus}\end{smallmatrix}\right) \oplus \underset{}{H_1(M_c)}\right)$$

$\langle \mu \rangle$ $\parallel$

$(1,0) \longmapsto (0,P) \oplus 0$

$(0,1) \longmapsto 0 \oplus (1,*)$

so $|\text{Tors}(H_1(M_\alpha))| = |H_1(M_k)| = |\det(\text{map})| = P$

<u>Thurston</u> (Hyperbolic Dehn filling)

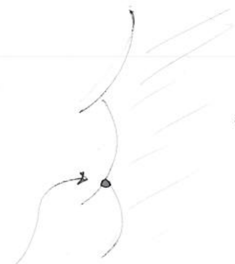
$P \gg 1$ M : hyp

$\Longrightarrow$ M_α : hyp.

and $\text{Vol}(M) \geq \text{vol}(M_\alpha)$.

so no hope to have result in $d = 3$

Zoom in on the frontier between thick and thin parts in universal cover

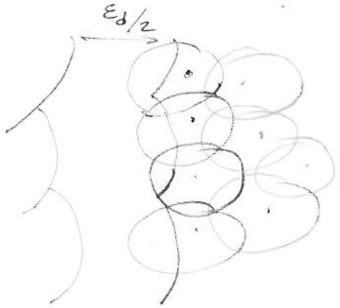


have corners.

Margulis constant

$$\widetilde{M}_{thin} = \bigcup \{ x : d(x, \gamma x) < \varepsilon_d \}$$
$$\gamma \in \Gamma = \pi_1 M$$

Repeat nerve construction here.



$\widetilde{M}_{thin}$ $\widetilde{M}_{thick}$

$\widetilde{M}_0$

$\varepsilon_d/2$ away from $\widetilde{M}_0$

Can show that $\exists$ (π_1-equivariant) deformation retract of $\widetilde{M}_{thick}$ onto (union of balls w/ centers) $=: \mathcal{U}$

$$\widetilde{M}_{thick} \xrightarrow[\text{retr.}]{\text{def}} \mathcal{U}$$

fails to work

naively, would like to use a vector field s.t. $\vec{V}(x) \in$ convex cone via gradients.

need $\forall\ \gamma \in \Gamma = \pi_1 M$.

pick $\varepsilon_\gamma < \varepsilon_d$

s.t. angles b/w translations

always acute.

* "Everything goes wrong in dim = 3 *.