MSRI - CAT(0)

Petra Schwer @KIT (slide and chalk talk)

Geometric methods for Deligne Lusztig varieties
w/ Elizabeth Milićević + Anne Thomas.

Affine Deligne Lusztig variety:

$X_x(b)$ = alg. variety $\subset G/I$

indexed by $x \in W$ = Weyl (G).

and $b \in G$.

Q: When is $X_x(b)$ non-empty?

Q: Under these circumstances, what is dimension?

Notation:

$G \supset B \supset T$

G: spl. conn. reductive; B: Borel subgp.; T: maximal torus.

(Obtain building)

eg $G = SL_3$ (pick base field $\mathbb{F}_q$)

$B = \left\{ \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \right\}$ - fixes a max'l simplex CHAMBER

$T = \left\{ \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \right\}$ - fixes a sphere: convex + compact APARTMENT.

$k = \overline{\mathbb{F}_q}$ alg. closure

F := Laurent series in k = $k((t))$

with elements:

$\sum_{j \geq m} a_j t^j$, $\mathcal{O} = k[[t]]$ Power series over k.

Fact: $\pi: \mathcal{O} \longrightarrow k$ is a projection

$\sum_{j \geq 0} a_j t^j \longmapsto a_0$

the IWAHORI subgroup $I \subset G(F)$ is

$I := \pi^{-1}(B(k))$

I: stabilizer of the fundamental chamber in Bruhat-Tits building X assoc. to $G(F)$

$\pi^{-1}(B(k))$: stabilizer of some chamber.

G/I := affine flag variety

$\cong$ chambers in X.

Goal: understand pattern (see slides)

related to retraction of apartments in Schrver's Thesis.

Formally,

$$X_x(b) = \{ g \in \underbrace{G(F)/I}_{\text{chambers}} : g^{-1} b \sigma(g) \in \underbrace{IxI}\}$$

where σ is extension of Frobenius map $a \mapsto a^q$, $\mathbb{F}_q \to \mathbb{F}_q$ to G. by acting on components.

For Weyl group

$$W_0 = N_G(T(F))/T(F) \quad \underline{\text{spherical}}$$

implies $G = \bigsqcup_{w \in W_0} BwB$.

$$W = N_G(T(F))/T(\mathcal{O}) \quad \underline{\text{affine}}$$

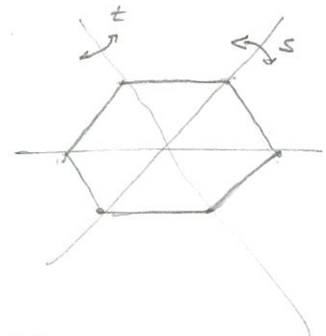
implies $G(F) = \bigsqcup_{w \in W} IwI$.

motivation for understanding $X_x(b)$

Example

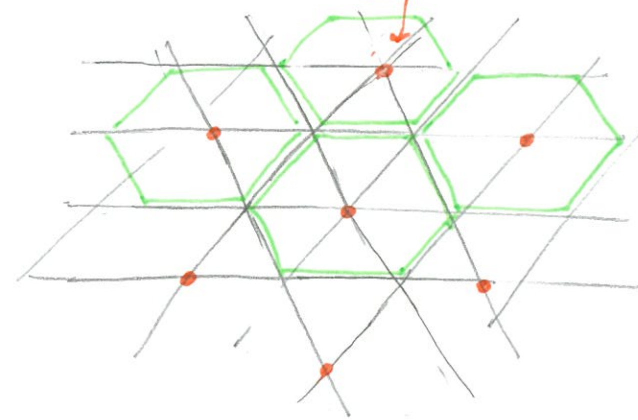
$G = \langle s, t \rangle$

(a Coxeter group)



$x \in W = W_0 \ltimes T$

↑ spherical ↑ root lattice



can obtain original picture of hexagonal - honeycomb.

⇝ each $x \in W$ has unique normal form

$t^{-1} \cdot w$

$\in T \quad \in W_0$

Fact : $\forall\ b \in G$

$\exists\ b' \in \tilde{W}$ s.t. b' is σ-conjugate to b

and

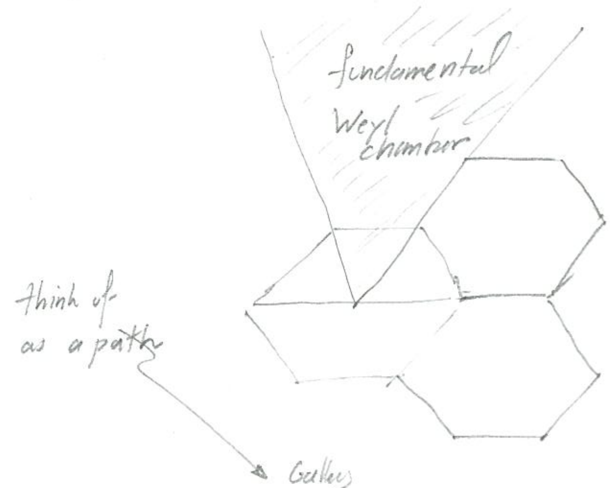
if b, b' are σ-conjugate

then $X_x(b) \cong X_x(b')$

in pictures gray Δ is σ-conjugate to black Δ

and pink Δ are NOT pairwise σ-conjugate

in example



if γ falls outside of chamber translate via reflection generators into fundamental chamber.

Conjecture (Görtz + Haines + Kottwitz + Reuman 2010)

$b = t^\lambda \in \tilde{W}$ a translation

$\Rightarrow\ \exists\ N_b \in \mathbb{N}$ s.t. $\forall\ x \in \tilde{W}$ and $l(x) > N_b$

$X_x(1_G)$ is nonempty $\Leftrightarrow$ $X_x(b)$ is nonempty

further

$$\dim X_x(b) = \dim\left(X_x(1_G) - \langle \rho, \lambda^+ \rangle\right)$$

Current work gives partial answer to this conjecture and extends some of the equivalences to those between elements that are σ-conjugate.

Milićević + Schwer + Thomas : Geometric approach (to conjecture)

Fix $b = t^\lambda$ a translation.

WTS:

(i) $X_x(b)$ nonempty $\iff$ $\exists$ "positive folded gallery" $\gamma : 1 \rightsquigarrow b$ of type x.

(ii) $\dim X_x(b)$ can be expressed in terms of (# positive folds) and (# positive crossings of γ)

(iii) construct + manipulate galleries γ while controlling folds and crossings.

In this setting a GALLERY is a sequence

$$\gamma = \left(c_0 \supset_{p_1} \subset c_1 \supset_{p_2} \subset c_2 \cdots \cdots c_n \right)$$

where c_i = maximal simplex,

p_i = "panels"

→ associated to some word.

a FOLD is some index i s.t. $c_{i-1} = c_i$.

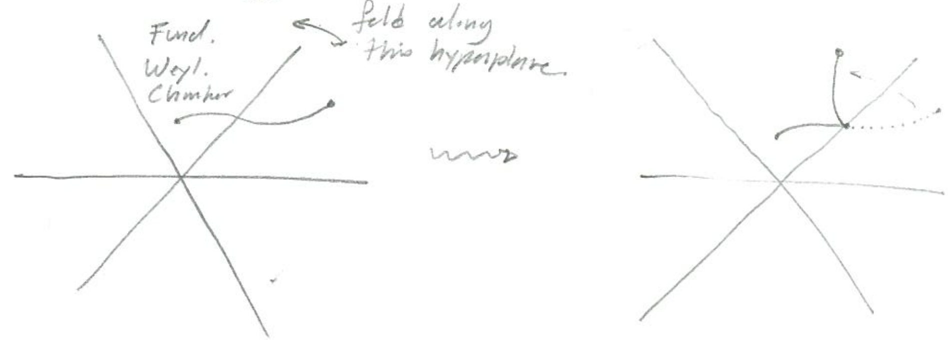
a fold is POSITIVE when $c_{i-1} = c_i$ is on "positive" side of the hyperplane spanned by p_i.

where positive is given by some consistent orientation of hyperplanes

→ think of "orientations" of ultrafilters in Sageev construction.

pick point ("sun") and direct each hyperplane towards said point ("face the sun")

Folding:



gallery ⇝ word in group.

fold ⇝ deleting generators
(decorate generators by deletion)

using representation theoretic tools
can then understand reflection length
in Coxeter group

independent work of Schwer.

N.B. these notes have omitted
diagrams + statements on slides.

Significance of Frobenius map.

- from strata in
Shimura variety

this is the map taking
one stratum to another.

Geometric methods for Affine Deligne Lusztig varieties

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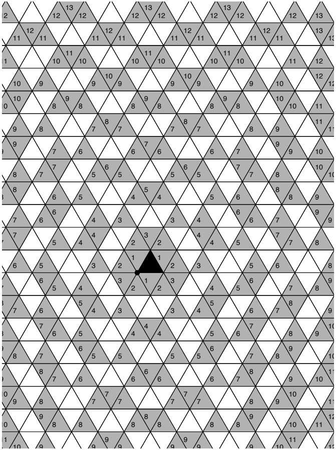
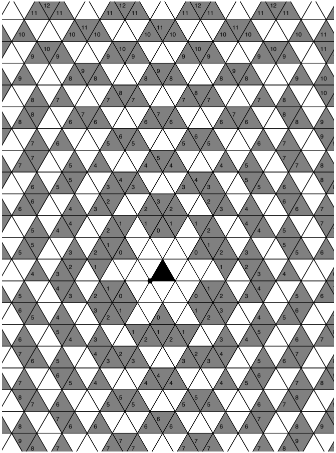
Karlsruhe Institute of Technology

Joint work with Elizabeth Milićević (Haverford College)
and Anne Thomas (University of Sydney)

MSRI Berkeley
September 27th 2016

Ultimate goal: explain these pictures





pictures by Görtz–Haines–Kottwitz–Reuman, arXiv:0504443

Main Questions

Nonemptiness: For which $(x, b) \in W \times W$ is $X_x(b) \neq \emptyset$?

Dimension: What is the dimension of $X_x(b)$?

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In case b is *basic* these questions are solved:

- Beazley=Milićević, Görtz-Haines-Kottwitz-Reuman, Reuman, Görtz-He, He, ...
- Görtz, He and Nie (2012):
  nonemptiness pattern for all x and all basic b
- He (Annals 2014):
  dimension formula for all x and basic b

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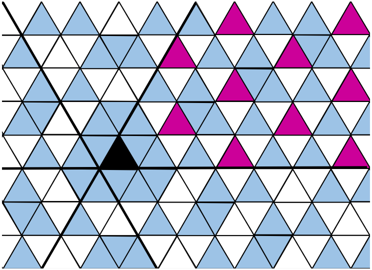
- Yang (2014): nonemptiness and dimension (x, b arbitrary) in case SL_3, that is type $\tilde{A}_2$.

The *basic* case

An element $b \in G(F)$ is basic if it is σ-conjugate to an element of length 0 in the extended affine Weyl group.

- All basic b in W are pairwise σ-conjugate.
- Dominant translations (pink) are not basic and pairwise not σ-conjugate.



basic elements (blue); translations in the dominant Weyl chamber (pink)

Conjectural nonemptiness and dimensions

The following is a special case of the conjecture for arbitrary pairs x, b.

Conjecture (Görtz-Haines-Kottwitz-Reuman 2010)

Let $b = t^\lambda \in W$ be a translation. There exists $N_b \in \mathbb{N}$ such that for all $x \in W$ with $\ell(x) > N_b$

$$X_x(1) \neq \emptyset \Longleftrightarrow X_x(b) \neq \emptyset,$$

and if both varieties are nonempty then

$$\dim X_x(b) = \dim X_x(1) - \langle \rho, \lambda^+ \rangle.$$

Our approach

In the following let $b = t^\lambda$ be a translation in W.
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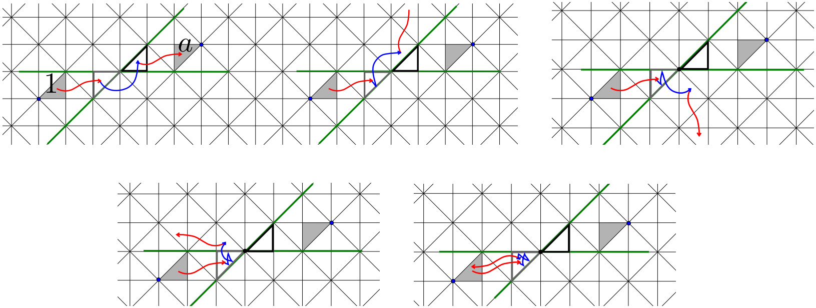
Again generalizing Gaussent-Littelmann/Parkinson-Ram-C.Schwer

(3) Construct and manipulate such galleries using root operators, combinatorics in Coxeter complexes and explicit transformations.

Explicit construction of a folded gallery

Construct a gallery $\gamma : 1 \rightsquigarrow 1$ of type $a = t^{2\rho} w_0$ as follows:



The gallery γ is positively folded for the right choice of a “sun”.

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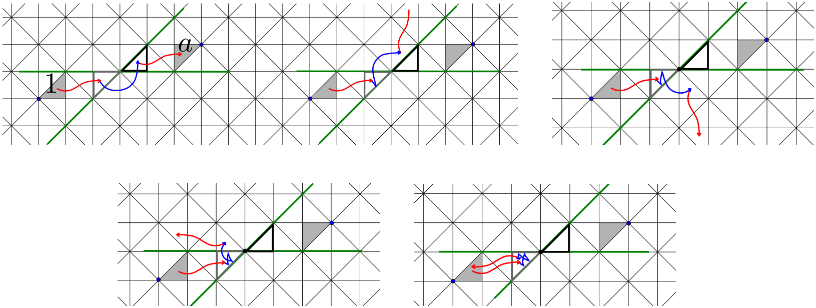
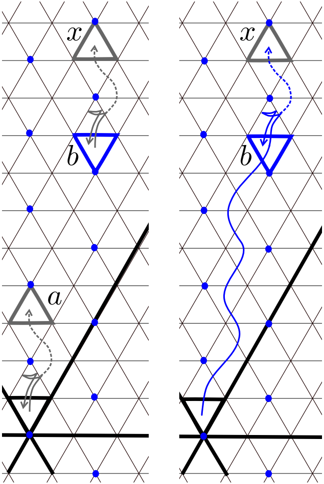


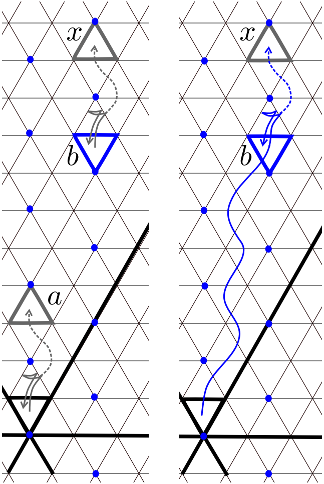
The gallery γ is positively folded for the right choice of a "sun".

We obtain: $X_a(1) \neq \emptyset$ and of dimension ≥ 7.

Manipulation of folded galleries



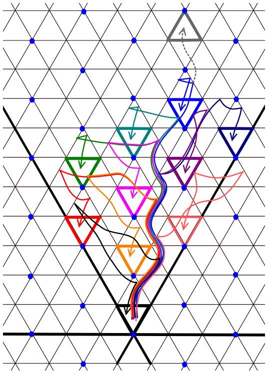
Manipulation of folded galleries



This implies $X_x(b) \neq \emptyset$ for $b = t^\mu$ dominant and close to x.

Root operators

Apply available root operators to the gallery constructed on the previous slide:



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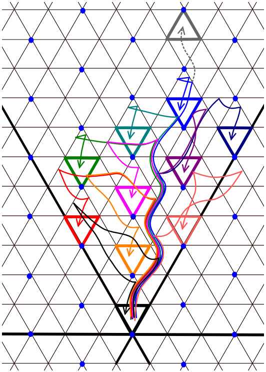


This implies $X_x(b) \neq \emptyset$ for most $b = t^\mu$ between 1 and x.

Theorem 1 (Milićević–S–Thomas)

Let $b = t^\mu$ be a pure translation and let $x = t^\lambda w \in W$. Assume that b is in the convex hull of x and the base alcove + two technical conditions on μ and λ. Then

$$X_x(1) \neq \emptyset \implies X_x(b) \neq \emptyset$$

and if $w = w_0$ then $X_x(1) \neq \emptyset$ and $X_x(b) \neq \emptyset$. If both varieties are nonempty then

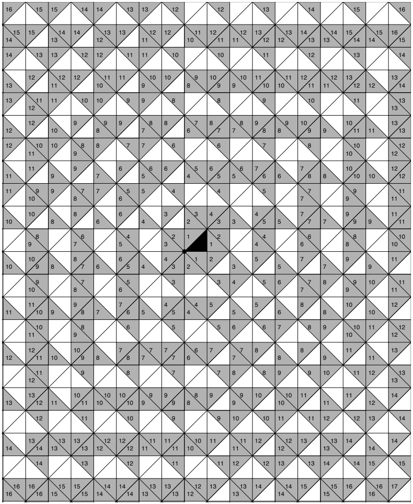
$$\dim X_x(b) = \dim X_x(1) - \langle \rho, \mu^+ \rangle.$$

Precise assumptions:

- $t^\lambda w_0$ and $t^{-\mu} x$ are in the shrunken dominant Weyl chamber $\widetilde{\mathcal{C}}_f$
- b is in the convex hull of x and the base alcove
- μ lies in the negative cone based at $\lambda - 2\rho$.

Theorem 1 in type $\tilde{C}_2$

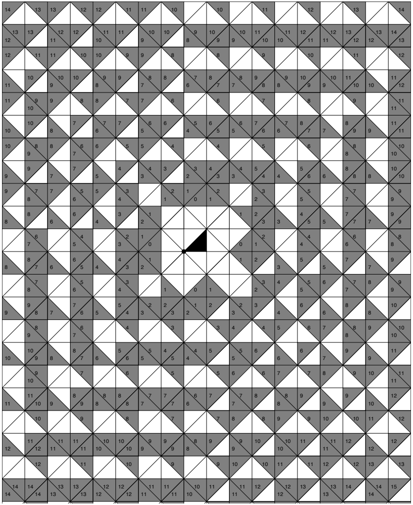
$b = 1$



picture: Görtz, Haines, Kottwitz and Reuman (2006)

Theorem 1 in type $\tilde{C}_2$

$b = t^{(1,0)}$



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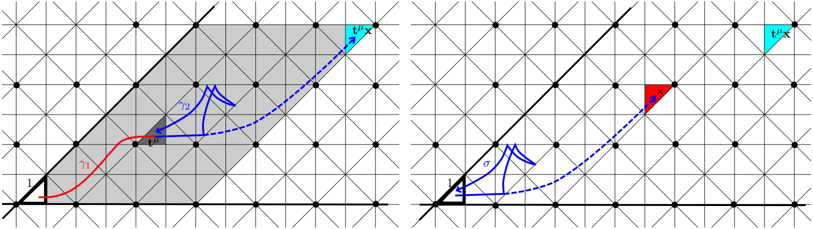
Geometric transformations of galleries



$$X_{t^\mu x}(t^\mu) \neq \emptyset \qquad \Longleftarrow \qquad X_x(1) \neq \emptyset$$

Theorem 2 (Milićević–S–Thomas)

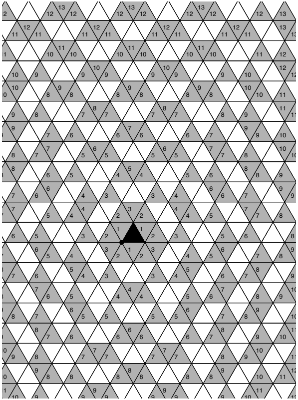
If $b = t^\mu$ *is in the convex hull of* $t^\mu x$ *and the base alcove, then*

$$X_x(1) \neq \emptyset \implies X_{t^\mu x}(t^\mu) \neq \emptyset$$

and if nonempty then $\dim X_{t^\mu x}(t^\mu) \geq \dim X_x(1)$.

What else?

We can show more about nonemptiness and dimensions:



- nonemptiness pattern of $\mathcal{C}_f$ transfers to other Weyl chambers (via conjugation)
- pattern is symmetric under diagram automorphisms.
- everything we prove holds in the p-adic setting.

Thank you!

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Our preprint is available at `arxiv:1504.07076`.

More results

Conjugation

Theorem 3 (Milićević–S–Thomas)

Let $b = t^\mu$ be a dominant pure translation and let $x = t^\lambda w \in W$. Assume that $t^\lambda w_0$ lies in the shrunken dominant Weyl chamber. Then for all $u \in W_0$

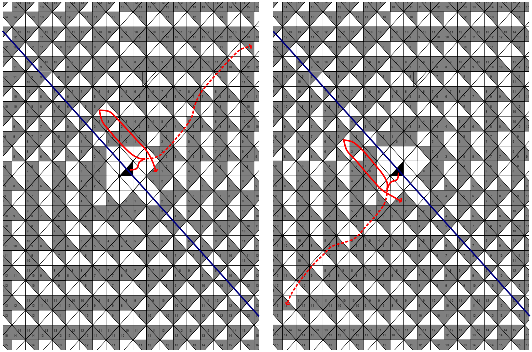
$$X_x(b) \neq \emptyset \implies X_{u^{-1}xu}(b) \neq \emptyset.$$

Moreover if these varieties are nonempty then

$$\dim X_{u^{-1}xu}(b) \geq \dim X_x(b) - \frac{1}{2}(\ell(u^{-1}xu) - \ell(x)).$$

Diagram automorphisms

Let g be an automorphism of the apartment induced by an automorphism of the diagram.



Theorem 4

$X_x(b) \neq \emptyset \Longleftrightarrow X_{g(x)}(g(b)) \neq \emptyset.$

If both are not empty they have the same dimension.

Arbitrary translation alcoves

Theorem 5 (Milićević–S–Thomas)

Let $b = t^{\mu}$ be a pure translation and let $x \in W$. Assume that

- *b is in the convex hull of x and the base alcove*
- *x and $t^{-\mu}x$ lie in the same Weyl chamber*
- *if x is in a shrunken Weyl chamber then $t^{-\mu}x$ is in a shrunken Weyl chamber*

Then

$$X_x(1) \neq \emptyset \implies X_x(b) \neq \emptyset.$$

Moreover if these varieties are nonempty then

$$\dim X_x(b) \geq \dim X_x(1) - \langle \rho, \mu^+ \rangle - \langle \rho_{B^-}, \mu + \mu_{B^-} \rangle.$$

The p-adic setting

Theorem 6 (Milićević–S–Thomas)

Let b be a translation and let $x \in W$. There is a reasonable combinatorial definition of $\dim X_x(b)_{\mathbb{Q}_p}$, and using this definition

$$\dim X_x(b) = \dim X_x(b)_{\mathbb{Q}_p}.$$

Corollary

All previous Theorems hold in the p-adic setting.

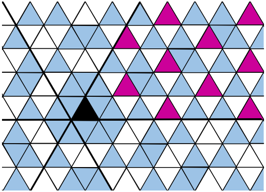
Basic case, previous results, conjectures

Previous results

Almost all previous results are in the basic case.

Common approach in the basic case:

- generalisation of classical Deligne–Lusztig theory,
- combinatorics on minimal length elements in conjugacy classes in the affine Weyl group W.

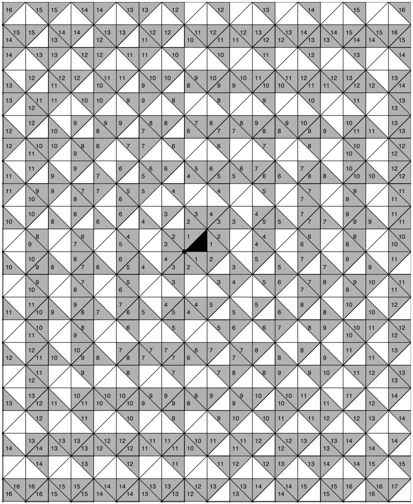


basic elements (blue); translations in the dominant Weyl chamber (pink)

- Beazley, Görtz, He, Haines, Kottwitz, Nie, Reuman, ...
- Görtz, He and Nie (2012): nonemptiness pattern $\forall x$ and $\forall b$ basic
- He (Annals 2014): dimension formula $\forall x$ and $\forall b$ basic

Computer experiments

In $G(F)/I$, Görtz, Haines, Kottwitz and Reuman (2006) conducted computer experiments in low rank; *e.g.* for $b = 1$:



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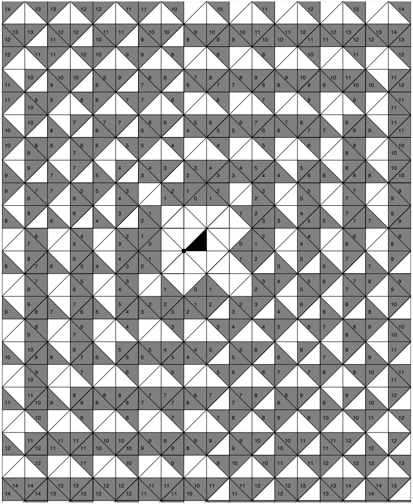


Conjecture for arbitrary b

Conjecture (Görtz-Haines-Kottwitz-Reuman 2010)

Let $b \in G(F)$. Then there exists $N_b \in \mathbb{N}$ such that for all $x \in \widetilde{W}$ with $\ell(x) > N_b$

$$X_x(b) \neq \emptyset \Longleftrightarrow X_x(\hat{b}) \neq \emptyset$$

and if both varieties are nonempty then

$$\dim X_x(b) = \dim X_x(\hat{b}) - \frac{1}{2}\left(\langle 2\rho, \nu_b\rangle + \mathrm{def}_G(b) - \mathrm{def}_G(\hat{b})\right).$$

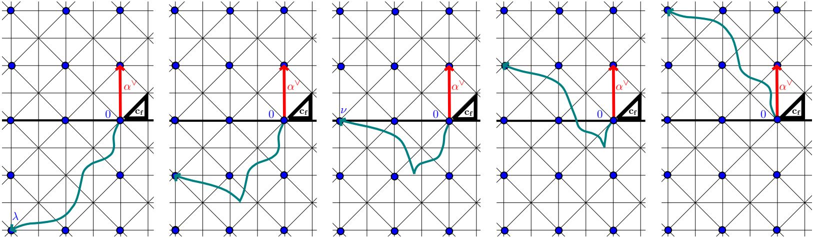
Here $\hat{b}$ is an associated basic element, def_G is the defect, and ν_b is the Newton point parameterizing the σ-conjugacy class.

- (Yang 2014) This conjecture holds for $G = SL_3$.

Root operators

Root operators

Root operators e_α, f_α for simple roots α were defined by Gaussent and Littelmann (2005). They act on sets of positively folded galleries of a fixed type.



$f_\alpha^2(\gamma)$ $\quad$ $f_\alpha(\gamma)$ $\quad$ γ $\quad$ $e_\alpha(\gamma)$ $\quad$ $e_\alpha^2(\gamma)$

Using properties of e_α and f_α we can easily control end-vertices and dimensions of galleries. If we work very carefully, we can also control start-alcoves and end-alcoves.
Availability of all root operators is crucial to our constructions.