MSRI - CAT(0)

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Hyperbolicity in CAT(0)-spaces
- Results
- Anosov Representations

Conjecture (Gromov): Every one-ended hyperbolic group contains a surface subgp.
$\pi_1(S_{g\geq 2})$

Progress:

Calegari + Walker: random groups are one-ended + hyperbolic] known previously and contain a quasi-convex surface subgroup.

undistorted.

* Seemingly hard to drop q.c. hypothesis even in non-random setting.

Kahn + Markovic (2012)

Γ: cocompact 3-orbifold ($<$ discrete $PSL(2,\mathbb{C})$) contains ∞-many conjugacy classes of quasiconvex surface subgps

again have this

Assumes this.

Perelman: (Geometrization 2002)

M: closed Riemannian 3-manifold of neg. curvature
$\Rightarrow \pi_1 M$ = cocompact lattice of $PSL(2,\mathbb{C})$

Agol, Wise (2011?) (Virtual Fibring)

Γ: cocompact lattice of $PSL(2,\mathbb{C})$
$\Rightarrow \Gamma$ contains ∞-many conjugacy classes of exponentially distorted surface subgroups
($\because$ fibre subgroups are necessarily exponentially distorted)

New: For G: simple Lie group of rank $=1$ and Γ a cocompact lattice.

$\longrightarrow$ analogous result with q.c. surface subgroups holds by:

Hamenstädt (2015)

$G \neq SO(2n,1)$, Γ as above

$\Rightarrow$ $\exists$ as many conjugacy classes of q.c. surface subgroups

- proof uses similar techniques to Kahn+Markovic.

Kahn + Hamenstädt

$G = SO(2n,1)$ is ok.

- proof is much more involved + fancy.

Kahn + Labourie + Mozes:

G: complex simple Lie group

Γ: cocompact lattice

$\Rightarrow$ same result.

Hamenstädt:

rank $=2$ simple Lie groups. (eg. $SO(2,3)$)

Rk: these surface subgps NOT totally geodesic boundary.

§2 Anosov representations

$\Sigma < PSL(2,\mathbb{C})$ a q.c. surface subgp.

$\mathcal{L}(\Sigma)$ LIMIT SET = accumulation points of $\Sigma \cdot x$ for $x \in \mathbb{H}^3$

$Hull(\mathcal{L}(\Gamma)) \subset \mathbb{H}^3$ is convex CONVEX HULL

$\Sigma \backslash Hull(\mathcal{L}(\Gamma))$ is compact.

$\longrightarrow$ convex cocompact

Example: $\mathbb{H}^2 \subset \mathbb{H}^3$ totally geodesic

$\Rightarrow$ $stab.(\mathbb{H}^2) < Isom(\mathbb{H}^3) = PSL(2,\mathbb{C})$

$stab.(\mathbb{H}^2) = PSL(2,\mathbb{R})$ cocompact surface subgp.

Limit set = round circle.

$\Gamma < PSL(2,\mathbb{C})$

$\forall$ round circles $S^1 \subset \partial\mathbb{H}^3$

$\exists$ sequence $\Sigma_i < \Gamma$. s.th.

$\mathcal{L}(\Sigma_i) \longrightarrow S^1$ in the Hausdorff topology

For concreteness: consider $G = SO(2,3)$ from now on.

$K = SO(2) \times SO(3)$

$q = x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2$ signature $(2,5)$

$X = G/K$

<u>Isotropic flags</u>: pairs $(\underset{\text{1-dim}}{v}, \underset{\text{2-dim.}}{W})$

(s.t. $x_2^2 - x_3^2 - x_4^2 - x_5^2 = 0$. ?)

W = 2-dim'l isotropic flags

$v \subset W$ is a line.

eg. $v = (1,0,1,0,0)$

$\hat{v} = (0,1,0,1,0)$

$W = \mathrm{span}(v, \hat{v})$

$\Longrightarrow (v, W)$ is an isotropic flag

G acts transitively on G/P — analogous to chambers of Weyl grp

$P = \mathrm{stab}(\text{1-flag})$

$=$ Borel subgrp. — minimal parabolic.

$\rightsquigarrow$ Furstenberg boundary

<u>Geometric description</u>

For $x \in X$

a direction $v \in T_x^u X$ is <u>REGULAR</u>

if it is tangent to a unique! totally geodesic <u>2-dim'l flat</u>

copy of $\mathbb{R}^2 \subset X$.

<u>barycentric direction</u> (unique in each chamber)

$G/P \sim$ barycentric directions from a given point.

Points $x, y \in G/P$ are called

<u>OPPOSITE</u> if $\exists$ geodesic connecting them.

"visual points" — X contains flats

$\Rightarrow$ not a visual space,

so only want pairs that are opposite

Due to: (Labourie 2006
Guichard + Wienhard 2012
Guéritaud + Guichard + Kassel 2015
Kapovich + Leeb + Porti 15)

$\Sigma < G$ is called <u>Anosov surface subgroup</u>

when $\partial\Sigma \hookrightarrow G/P$ is Σ-equivariant embedding.

and distinct points $x, y \in \partial\Sigma$

map to opposite points in G/P.

<u>Construction</u>:

For barycentric direction

+ one-parameter subgroup of G

obtain geodesic flow $G \curvearrowright T$

<u>Harish + Chandra + Moore + more</u>

$\exists$ periodic (closed) geodesic γ

contracting? directions w/ controlled endpts.

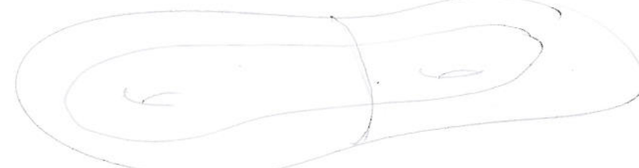
$\Rightarrow$ obtain entire torus foliated by geodesic, totally geodesic + immersed

since any direction has another arbitrarily close direction.

Lifting γ to $\tilde{\gamma} \subset X$

totally geodesic embedded hyperbolic plane $\mathbb{H}^2 \subset X$.

containing $\tilde{\gamma}$



totally geodesic torus $\to$ allows gluing pants along F

Higher rank: need fancy stuff